Correlations in Lévy Interest Rate Models

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1 Introduction

Studying the dynamic development of historical yield curves one sees three basic movements: parallel shifts, twists, and changes of concavity or convexity. They occur in any combination. The complexity of these movements corresponds to a wide range of correlation patterns or covariance structures of the various interest rates. In simple one-factor short rate models the evolution of the whole yield curve is completely determined by the evolution of a single rate. This means that there is perfect correlation along the curve. A market shock to interest rates affects all maturities in the same way. Nevertheless, a simple short rate model may be useful to price a single product which does not depend on the joint distribution of several interest rates. However, often a whole range of financial products across many different maturities has to be priced consistently within one model. Then the joint behaviour of the interest rates does matter. With the increased volatility of the interest rate slope in recent years, correlation derivatives products have become more strategic. We mention just CMS spread options.

Extensions of the one-factor to a two- or three-factor model already allow to describe the empirically observed correlations between short-, middle-, and long-term maturities in a more realistic way. It was in the HJM forward rate approach ([12]) where the full range of maturities could be considered simultaneously. This change to a function valued view has made it possible to include the current yield curve in the model. The basic quantities in the HJM approach are the instantaneous forward rates. Since these rates are somewhat artificial, it has finally been the BGM or LIBOR market model which has become very popular among practitioners. Volatility surfaces from the actual markets show (see e.g. Eberlein and Kluge [6]) that the Brownian motion driven HJM and market models are still not flexible enough to describe the real markets with sufficient accuracy. In a series of papers a Lévy process driven interest rate theory has been developed in order to achieve sufficient flexibility ([10, 8, 4, 6, 7, 5]). During the extensions of the initial Lévy forward and Lévy LIBOR model it turned out that the natural driving processes for interest rate models are time-inhomogeneous Lévy processes. On this level the forward rate model, the LIBOR model, and the forward process model are briefly described in the following. The interest for the latter stems from its computational efficiency. For the implementation of these models a weak form of time-inhomogeneity is sufficient, namely piecewise time-homogeneous Lévy processes.

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The purpose of this paper is to derive explicit formulae for the correlations in all three models and to show the effects of the variation of the parameters of the generating distribution on the correlation structure. A typical result is visualized in Figure 2 which shows the correlations of zero coupon bond prices with maturities of two and four years and five and ten years, respectively. The correlations increase if the driving process approaches a Brownian motion which can be approximated by choosing $\alpha = \delta = 1000$. In section 6 the extension of the LIBOR (and analogously of the forward process) model to a multicurrency setting is considered. Explicit formulae for the correlations of the interest rates across different economies are given.

In the last section we calibrate the Lévy forward model using empirical correlations which are derived from historical prices of government bonds. The underlying data set was provided by the Deutsche Bundesbank. The comparison of the Lévy driven model with the classical Brownian motion driven model shows clearly that the latter one is not able to describe empirically observed correlations appropriately.

2 The driving process

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete stochastic basis, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$ satisfies the usual conditions. $T^* > 0$ is a finite time horizon and $\mathcal{F} = \mathcal{F}_{T^*}$. The driving process $L = (L_t)_{t\in[0,T^*]}$ is a d-dimensional *time-inhomogeneous Lévy process*, i.e. an adapted process with independent increments and absolutely continuous characteristics. The law of L_t is given by its characteristic function

$$\mathbb{E}\left[e^{i\langle u,L_t\rangle}\right] = \exp\left(\int_0^t \left(i\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle\right) F_s(dx)\right) ds\right). \tag{1}$$

As a consequence of assumption ($\mathbb{EM}$) which is specified below, L_t has finite expectation for all $t \in [0, T^*]$. Therefore a truncation function is not needed in (1). Here, $b_s \in \mathbb{R}^d$ is the drift term, c_s, the quadratic variation of the diffusion part, is a symmetric non-negative definite $d\times d$-matrix, and $F_s(dx)$, the Lévy measure, is a measure on $\mathbb{R}^d$ which integrates $(x^2 \wedge 1)$ and satisfies $F_s(\{0\}) = 0$. We denote the Euclidian scalar product on $\mathbb{R}^d$ by $\langle\cdot,\cdot\rangle$ and the respective norm by $|\cdot|$. We shall assume that

$$\int_0^{T^*} \left(|b_s| + \|c_s\| + \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\, F_s(dx)\right) ds < \infty, \tag{2}$$

where $\|c_s\| = \sup_{|x|\leq 1} |c_s x|$ is the usual norm on the $d\times d$-matrices. We can also assume that L starts from 0 and that the paths are right-continuous and have left-hand limits. Furthermore, we shall make the following assumption on exponential moments.

Assumption ($\mathbb{EM}$): *There are constants M, $\varepsilon > 0$ such that*

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp\langle u, x\rangle\, F_s(dx)\, ds < \infty$$

for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$. Without loss of generality we can also assume that $\int_{\{|x|>1\}} \exp\langle u, x\rangle\, F_s(dx) < \infty$ for all $s \in [0, T^*]$.

Assumption ($\mathbb{EM}$) is equivalent to $\mathbb{E}[\exp\langle u, L_t\rangle] < \infty$ for all $t \in [0, T^*]$ and all u as stated above. In view of the martingale property of the underlying processes, the latter is a very natural assumption for the interest rate models which we shall consider later. Under ($\mathbb{EM}$) L is a special semimartingale with canonical representation

$$L_t = \int_0^t b_s\,ds + \int_0^t \sqrt{c_s}\,dW_s + \int_0^t \int_{\mathbb{R}^d} x\,(\mu - \nu)(ds, dx) \tag{3}$$

for all $t \in [0, T^*]$, where W is a standard d-dimensional Brownian motion and $\sqrt{c_s}$ is a measurable version of the square root of c_s. μ is the random measure of jumps of L with compensator $\nu(ds, dx) = F_s(dx)\,ds$.

Define

$$\theta_s(z) := \langle z, b_s \rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z, x\rangle} - 1 - \langle z, x\rangle \right) F_s(dx) \tag{4}$$

for $s \in [0, T^*]$ and $z \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$, where M is the constant from assumption ($\mathbb{EM}$). The following theorem is a special case of Proposition 8 in Eberlein and Kluge [6]. See Kluge [13, Proposition 1.9] for a proof in this generality. For (time-homogeneous) Lévy processes this theorem goes back to Eberlein and Raible [10]. The result is fundamental for the calculation of the correlations in the following sections and it is also needed to construct the Lévy forward rate model in section 3.

Theorem 1 *Let $0 \leq t < T \leq T^*$ and suppose that $f : \mathbb{R}^{\geq 0} \to [-M, M]^d$ is a continuous function. Then*

$$\mathbb{E}\left[\exp\left(\int_t^T f(s)\,dL_s \right) \right] = \exp\left(\int_t^T \theta_s(f(s))\,ds \right).$$

3 The Lévy forward rate model

The Lévy forward rate model has been introduced in Eberlein and Raible [10] in a risk-neutral setting and it has been extended to the general setting in [8]. Results on the uniqueness of the risk-neutral measure can be found in [4]. See [6] and [5] for the calibration of the model and for efficient pricing of interest rate derivatives. The basic quantities to be modeled in the Lévy HJM approach are the instantaneous forward rates $f(t, T)$ given in the form

$$f(t, T) := f(0, T) + \int_0^t \alpha(s, T)\,ds - \int_0^t \sigma(s, T)\,dL_s, \tag{5}$$

where $0 \leq t \leq T \leq T^*$, L is a d-dimensional time-inhomogeneous Lévy process and $f(0, T)$ are the given initial values. The coefficients $\alpha : [0, T^*] \times [0, T^*] \to \mathbb{R}$ and $\sigma : [0, T^*] \times [0, T^*] \to \mathbb{R}^d$ are deterministic, bounded functions such that $0 \leq \int_t^T \sigma^i(t, s)\,ds \leq M/2$ for all $t \leq T$ and all $i \in \{1, \ldots, d\}$. M is the constant from assumption ($\mathbb{EM}$). Furthermore, we assume that $\alpha(s, T) = 0$ and $\sigma(s, T) = 0$ for $s > T$. In general, it is possible to assume that $\alpha(t, T)$ and $\sigma(t, T)$ are random (see [4, 8]). But for the implementations we choose deterministic volatility structures and therefore we make this assumption here from the very beginning. Define $\Sigma(s, T) := \int_{s \wedge T}^T \sigma(s, u)\,du$ and $\Sigma(s, t, T) := \Sigma(s, T) - \Sigma(s, t)$. Recall that the price of a (default-free) zero coupon bond with maturity T at time $t \leq T$ is given by

$$B(t, T) = \exp\left(-\int_t^T f(t, u)\,du \right).$$

Let $A(s,T) := \int_{s\wedge T}^{T} \alpha(s,u)\,du$ and $A(s,t,T) := A(s,T) - A(s,t)$. Essentially by applying Fubini's theorem, we obtain

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left(-\int_0^t A(s,t,T)\,ds + \int_0^t \Sigma(s,t,T)\,dL_s \right), \tag{6}$$

where the starting values $B(0,t) \neq 0$ and $B(0,T) \neq 0$ are given. Let $r(s) := f(s,s)$ denote the short rate. Again by Fubini's theorem, the risk-free savings account $B_t := \exp\left(\int_0^t r(s)\,ds\right)$ can be written as

$$B_t = \frac{1}{B(0,t)} \exp\left(\int_0^t A(s,t)\,ds - \int_0^t \Sigma(s,t)\,dL_s \right). \tag{7}$$

From (6) and (7) we get another useful representation of zero coupon bond prices

$$B(t,T) = B(0,T) \exp\left(\int_0^t (r(s) - A(s,T))\,ds + \int_0^t \Sigma(s,T)\,dL_s \right). \tag{8}$$

Now, we choose $A(s,T) := \theta_s(\Sigma(s,T))$. Then Theorem 1 and (8) yield that the discounted bond price processes $(B_t^{-1}B(t,T))_{t\in[0,T]}$ are martingales for every $T \in [0,T^*]$. Thus, the market is arbitrage-free.

In the Lévy forward rate model the forward processes $F(\cdot,T_1,T_2)$ and the LIBOR rates $L(t,T)$ for an investment over a time period of length δ beginning in T can be obtained from (6) using

$$F(t,T_1,T_2) = \frac{B(t,T_1)}{B(t,T_2)} \quad \text{and} \quad L(t,T) = \frac{1}{\delta}(F(t,T,T+\delta) - 1). \tag{9}$$

3.1 The correlations of zero coupon bond prices

In this part, we study the correlations of zero coupon bond prices in the Lévy forward rate model.

Theorem 2 *Let $0 \leq t \leq T_1 \leq T_2 \leq T^*$. The correlation of $B(t,T_1)$ and $B(t,T_2)$ is*

$$\mathrm{Corr}(B(t,T_1), B(t,T_2)) = \frac{g_1(t,T_1,T_2) - g_2(t,T_1,T_2)}{\sqrt{h(t,T_1)}\sqrt{h(t,T_2)}},$$

where

$$\begin{aligned} g_1(t,T_1,T_2) &:= \exp\left(\int_0^t \theta_s\Big(\Sigma(s,t,T_1) + \Sigma(s,t,T_2)\Big)\,ds \right), \\ g_2(t,T_1,T_2) &:= \exp\left(\int_0^t \Big(\theta_s(\Sigma(s,t,T_1)) + \theta_s(\Sigma(s,t,T_2))\Big)\,ds \right) \end{aligned}$$

and

$$h(t,T) := \exp\left(\int_0^t \theta_s(2\Sigma(s,t,T))\,ds \right) - \exp\left(\int_0^t 2\theta_s(\Sigma(s,t,T))\,ds \right). \tag{10}$$

Proof: From (6) and Theorem 1 we obtain

$$\mathbb{E}[B(t,T)] = \frac{B(0,T)}{B(0,t)} \exp\left(-\int_0^t A(s,t,T)\,ds \right) \exp\left(\int_0^t \theta_s(\Sigma(s,t,T))\,ds \right)$$

and

$$\mathbb{E}\left[B(t,T)^2\right] = \frac{B(0,T)^2}{B(0,t)^2}\exp\left(-2\int_0^t A(s,t,T)\,ds\right)\exp\left(\int_0^t \theta_s(2\Sigma(s,t,T))\,ds\right).$$

Hence, we have

$$\mathrm{Var}(B(t,T)) = h(t,T)\frac{B(0,T)^2}{B(0,t)^2}\exp\left(-2\int_0^t A(s,t,T)\,ds\right).$$

Similarly we get

$$\begin{aligned}&\mathbb{E}[B(t,T_1)B(t,T_2)]\\&\quad = g_1(t,T_1,T_2)\frac{B(0,T_1)B(0,T_2)}{B(0,t)^2}\exp\left(-\int_0^t (A(s,t,T_1)+A(s,t,T_2))\,ds\right)\end{aligned}$$

and

$$\mathrm{Corr}(B(t,T_1),B(t,T_2)) = \frac{\mathbb{E}[B(t,T_1)B(t,T_2)] - \mathbb{E}[B(t,T_1)]\mathbb{E}[B(t,T_2)]}{\sqrt{\mathrm{Var}(B(t,T_1))}\sqrt{\mathrm{Var}(B(t,T_2))}}$$

completes the proof. □

In the same way, we obtain the correlation of $B(t_1,T_1)$ and $B(t_2,T_2)$ for different times t_1 and t_2. Let $0 \le t_1 \le T_1 \le T^*$ and $0 \le t_2 \le T_2 \le T^*$ such that $t_1 \le t_2$. Then the correlation of $B(t_1,T_1)$ and $B(t_2,T_2)$ is

$$\begin{aligned}&\mathrm{Corr}(B(t_1,T_1),B(t_2,T_2))\\&\quad = \exp\left(\int_{t_1}^{t_2}\theta_s(\Sigma(s,t_2,T_2))\,ds\right)\frac{g_1(t_1,t_2,T_1,T_2) - g_2(t_1,t_2,T_1,T_2)}{\sqrt{h(t_1,T_1)}\sqrt{h(t_2,T_2)}},\end{aligned} \tag{11}$$

where

$$\begin{aligned}g_1(t_1,t_2,T_1,T_2) &:= \exp\left(\int_0^{t_1}\theta_s\Big(\Sigma(s,t_1,T_1)+\Sigma(s,t_2,T_2)\Big)\,ds\right),\\ g_2(t_1,t_2,T_1,T_2) &:= \exp\left(\int_0^{t_1}\Big(\theta_s(\Sigma(s,t_1,T_1))+\theta_s(\Sigma(s,t_2,T_2))\Big)\,ds\right)\end{aligned}$$

and h is given by (10).

Expressions for the correlations of the forward process and the LIBOR rates can be derived analogously starting from definition (9).

3.2 Generalized hyperbolic Lévy motion

Generalized hyperbolic Lévy motions allow an almost perfect fit of model returns to financial data (see Eberlein [3]). They are generated by generalized hyperbolic (GH) distributions, i.e. a class of distributions with parameters $\lambda \in \mathbb{R}$, $\alpha > 0$, $\beta \in (-\alpha,\alpha)$, $\delta > 0$ and $\mu \in \mathbb{R}$. Given the five parameters its characteristic function is

$$\Phi_{\mathrm{GH}}(u) = e^{\mathrm{i}\mu u}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+\mathrm{i}u)^2}\right)^{\frac{\lambda}{2}}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}, \tag{12}$$

where K_λ denotes the modified Bessel function of the third kind with index λ. Generalized hyperbolic distributions are infinitely divisible (see [1]). Therefore, they generate

Lévy processes which we call generalized hyperbolic Lévy motions. For $\lambda = -1/2$ one gets the subclass of normal inverse Gaussian (NIG) Lévy motions. The characteristic function of NIG distributions simplifies to

$$\Phi_{\mathrm{NIG}}(u) = e^{\mathrm{i}\mu u} \frac{\exp\left(\delta\sqrt{\alpha^2-\beta^2}\right)}{\exp\left(\delta\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2}\right)}. \tag{13}$$

For most purposes in financial modeling subclasses with a smaller number of parameters (such as the NIG distributions or the hyperbolic distributions which correspond to $\lambda = 1$) already provide sufficient flexibility.

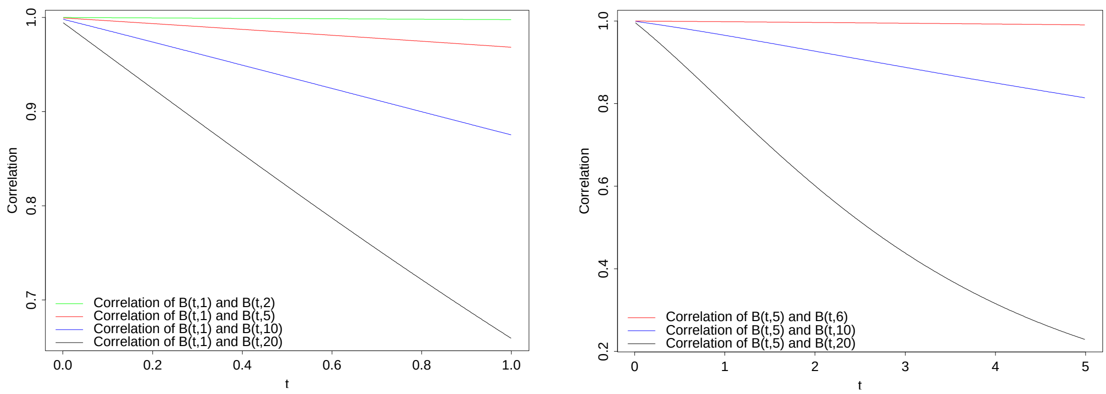

Figure 1: Correlation of zero coupon bond prices for $\alpha = 100$, $\beta = 0$, $\delta = 1$ and $a = 0.05$

Figure 1 shows the correlations of zero coupon bond prices along the time axis t up to maturity of the bond with the shorter maturity for different pairs T_1, T_2. The driving process L is a NIG Lévy motion and we used a Vasiček volatility structure

$$\sigma(s,T) = \sigma_0 e^{-a(T-s)} \tag{14}$$

with $a \neq 0$. We can set $\sigma_0 = 1$ as this parameter can be included in the driving Lévy process. Note that the correlation does not depend on the parameter μ.

The normal inverse Gaussian distribution with $\beta = 0$ converges weakly to a normal distribution with variance σ^2 if $\alpha \to \infty$ and $\delta \to \infty$ such that $\delta/\alpha \to \sigma^2$ (see Eberlein and von Hammerstein [11]). Figure 2 shows that the correlations of zero coupon bond prices increase if the driving process approaches a standard Brownian motion.

4 The Lévy LIBOR model

In the Lévy LIBOR model (or Lévy market model) the basic quantities are the forward LIBOR rates which are modeled directly by a driving time-inhomogeneous Lévy process.

4.1 The construction of the model

The Lévy LIBOR model is constructed via a backward induction (see Eberlein and Özkan [9]). Assume that a discrete tenor structure $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = T^*$ is given and set $\delta_k := T_{k+1} - T_k$. The starting point is a d-dimensional time-inhomogeneous Lévy process $L^{T_{N+1}}$ on a complete stochastic basis $(\Omega, \mathcal{F}_{T_{N+1}}, \mathbb{F}, \mathbb{P}_{T_{N+1}})$

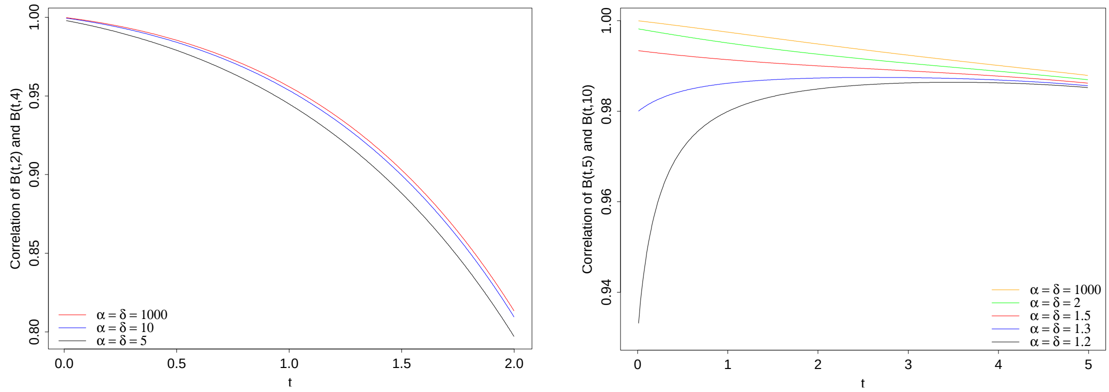

Figure 2: Correlation of zero coupon bond prices for $\beta = 0$, $a = 0.7$ (on the left side) and $\beta = 0$, $a = 0.005$ (on the right side respectively)

which satisfies the assumption ($\mathbb{EM}$). $\mathbb{P}_{T_{n+1}}$ can be interpreted as the forward measure corresponding to time T_{N+1}. According to (3) the canonical representation of $L^{T_{N+1}}$ is

$$L_t^{T_{N+1}} = \int_0^t b_s^{T_{N+1}}\, ds + \int_0^t \sqrt{c_s}\, dW_s^{T_{N+1}} + \int_0^t \int_{\mathbb{R}^d} x\, (\mu - \nu^{T_{N+1}})(ds, dx), \tag{15}$$

where $W^{T_{N+1}}$ is a d-dimensional standard Brownian motion under $\mathbb{P}_{T_{N+1}}$. μ is the random measure of jumps of $L^{T_{N+1}}$ and $\nu^{T_{N+1}}(ds, dx) = F_s^{T_{N+1}}(dx)ds$ is the $\mathbb{P}_{T_{N+1}}$-compensator of μ. The model is based on the following assumptions.

Assumption ($\mathbb{LR}$.1): *For every T_i there is a bounded, continuous, deterministic function $\lambda(\cdot, T_i) : [0, T^*] \to \mathbb{R}^d$ which represents the volatility of the forward LIBOR rate $L(\cdot, T_i)$. We assume that*

$$\sum_{i=1}^{N} |\lambda^j(s, T_i)| \leq M$$

for all $s \in [0, T^*]$ and all $j \in \{1, \ldots, d\}$. M is the constant from assumption ($\mathbb{EM}$) and $\lambda(s, T_i) = 0$ for $s > T_i$.

Assumption ($\mathbb{LR}$.2): *The initial zero coupon bond prices $B(0, T_i)$, $i \in \{1, \ldots, N+1\}$, are strictly positive and strictly decreasing in i. Consequently, the initial values of the LIBOR rates are given by*

$$L(0, T_i) = \frac{1}{\delta_i} \left(\frac{B(0, T_i)}{B(0, T_{i+1})} - 1 \right) > 0.$$

To start the backward induction the most distant LIBOR rate $L(t, T_N)$ is defined as

$$L(t, T_N) := L(0, T_N) \exp\left(\int_0^t \lambda(s, T_N)\, dL_s^{T_{N+1}} \right). \tag{16}$$

The drift coefficient $b^{T_{N+1}}$ is chosen in such a way that $L(\cdot, T_N)$ is a martingale under $\mathbb{P}_{T_{N+1}}$. Then, the forward measure $\mathbb{P}_{T_N}$ corresponding to time T_N is defined by the Radon–Nikodym derivative

$$\frac{d\mathbb{P}_{T_N}}{d\mathbb{P}_{T_{N+1}}} = \frac{1 + \delta_N L(T_N, T_N)}{1 + \delta_N L(0, T_N)}. \tag{17}$$

Applying Girsanov's theorem (for semimartingales), we see that under $\mathbb{P}_{T_N}$ the process $L^{T_{N+1}}$ is a semimartingale with canonical representation

$$L_t^{T_{N+1}} = \int_0^t \tilde{b}_s^{T_N}\, ds + \int_0^t \sqrt{c_s}\, dW_s^{T_N} + \int_0^t \int_{\mathbb{R}^d} x\, (\mu - \nu^{T_N})(ds, dx), \tag{18}$$

where W^{T_N} is a standard Brownian motion. ν^{T_N} is the $\mathbb{P}_{T_N}$-compensator of μ. In general, ν^{T_N} and $\tilde{b}_s^{T_N}$ are no longer deterministic and therefore $L^{T_{N+1}}$ is not a time-inhomogeneous Lévy process under $\mathbb{P}_{T_N}$ anymore. In the next step the LIBOR rates $L(\cdot, T_{N-1})$ are defined. For a drift term $(b_s^{T_N})$ consider the semimartingale

$$L_t^{T_N} := \int_0^t b_s^{T_N}\, ds + \int_0^t \sqrt{c_s}\, dW_s^{T_N} + \int_0^t \int_{\mathbb{R}^d} x\, (\mu - \nu^{T_N})(ds, dx), \tag{19}$$

where c_s, W^{T_N} and ν^{T_N} are as in (18). The LIBOR rates $L(t, T_{N-1})$ are given by

$$L(t, T_{N-1}) := L(0, T_{N-1}) \exp\left(\int_0^t \lambda(s, T_{N-1})\, dL_s^{T_N} \right). \tag{20}$$

Again, b^{T_N} is chosen in such a way that $L(\cdot, T_{N-1})$ is a martingale under $\mathbb{P}_{T_N}$.

Continuing via backward induction we get forward LIBOR rates $L(\cdot, T_j)$ and forward martingale measures $\mathbb{P}_{T_{j+1}}$ such that for every $j \in \{1, \dots, N\}$ $L(\cdot, T_j)$ is a $\mathbb{P}_{T_{j+1}}$-martingale of the form

$$L(t, T_j) = L(0, T_j) \exp\left(\int_0^t \lambda(s, T_j)\, dL_s^{T_{j+1}} \right), \tag{21}$$

where $L^{T_{j+1}}$ is a semimartingale with the canonical representation

$$L_t^{T_{j+1}} = \int_0^t b_s^{T_{j+1}}\, ds + \int_0^t \sqrt{c_s}\, dW_s^{T_{j+1}} + \int_0^t \int_{\mathbb{R}^d} x\, (\mu - \nu^{T_{j+1}})(ds, dx) \tag{22}$$

for an appropriately chosen drift term $(b_s^{T_{j+1}})$.

In (22), $W^{T_{j+1}}$ is a d-dimensional standard Brownian motion under $\mathbb{P}_{T_{j+1}}$ and $\nu^{T_{j+1}}(ds, dx)$ is the $\mathbb{P}_{T_{j+1}}$-compensator of μ. From the backward induction (see [9, p. 338–342]) we derive the relations

$$W_t^{T_j} = W_t^{T_{j+1}} - \int_0^t \ell(s-, T_j) \sqrt{c_s} \lambda(s, T_j)\, ds \tag{23}$$

where

$$\ell(s-, T_j) := \frac{\delta_j L(s-, T_j)}{1 + \delta_j L(s-, T_j)} \tag{24}$$

and

$$\nu^{T_{j+1}}(ds, dx) = F_s^{T_{j+1}}(dx) ds \tag{25}$$

where

$$F_s^{T_j}(dx) = \left(\ell(s-, T_j) \left(e^{\langle \lambda(s, T_j), x \rangle} - 1 \right) + 1 \right) F_s^{T_{j+1}}(dx). \tag{26}$$

Let $i > j$. Then $L^{T_{i+1}}$ and $L^{T_{j+1}}$ differ only by a drift term. We have

$$L_t^{T_{i+1}} - L_t^{T_{j+1}} = \int_0^t d_s^{i,j}\, ds \tag{27}$$

with

$$d_s^{i,j} = b_s^{T_{i+1}} - b_s^{T_{j+1}} + \sum_{k=j+1}^{i} \ell(s-, T_k) \left(c_s \lambda(s, T_k) + \int_{\mathbb{R}^d} x \left(e^{\langle \lambda(s,T_k), x \rangle} - 1 \right) F_s^{T_{k+1}}(dx) \right). \tag{28}$$

4.2 The correlations of the LIBOR rates

As already pointed out, the driving processes $L^{T_{j+1}}$, which are constructed during the backward induction, are in general not time-inhomogeneous Lévy processes anymore. From now on, we approximate the random terms

$$\ell(s-, T_i) = \frac{\delta_i L(s-, T_i)}{1 + \delta_i L(s-, T_i)}$$

by their deterministic initial values

$$\ell(0, T_i) = \frac{\delta_i L(0, T_i)}{1 + \delta_i L(0, T_i)}. \tag{29}$$

This will enable us to calculate the correlations of the LIBOR rates because then all driving processes $L^{T_{i+1}}$ are time-inhomogeneous Lévy processes and differ only by deterministic drift terms as given in (27). Set

$$\tilde{\theta}_s^{T_{i+1}}(z) := \frac{1}{2}\langle z, c_s z \rangle + \int_{\mathbb{R}^d} \left(e^{\langle z, x \rangle} - 1 - \langle z, x \rangle \right) F_s^{T_{i+1}}(dx) \tag{30}$$

and define the time-inhomogeneous Lévy process $\tilde{L}^{T_{i+1}}$ by $\tilde{L}_t^{T_{i+1}} := L_t^{T_{i+1}} - \int_0^t b_s^{T_{i+1}}\, ds$.

Theorem 3 *Let $i,\ j,\ k \in \{1, \ldots, N\}$ and $0 \le t \le \min\{T_i, T_j\}$. Then, given the measure $\mathbb{P}_{T_{k+1}}$ (and under the approximation), the correlation of the LIBOR rates $L(t, T_i)$ and $L(t, T_j)$ is*

$$\mathrm{Corr}_{\mathbb{P}_{T_{k+1}}}(L(t,T_i), L(t,T_j)) = \frac{g_1(t,i,j,k) - g_2(t,i,j,k)}{\sqrt{h(t,i,k))}\sqrt{h(t,j,k)}},$$

where

$$g_1(t,i,j,k)) := \exp\left(\int_0^t \tilde{\theta}_s^{T_{k+1}} \left(\lambda(s,T_i) + \lambda(s,T_j) \right) ds \right),$$
$$g_2(t,i,j,k)) := \exp\left(\int_0^t \left(\tilde{\theta}_s^{T_{k+1}} (\lambda(s,T_i)) + \tilde{\theta}_s^{T_{k+1}} (\lambda(s,T_j)) \right) ds \right)$$

and for $l \in \{i, j\}$ we set

$$h(t,l,k) := \exp\left(\int_0^t \tilde{\theta}_s^{T_{k+1}}(2\lambda(s,T_l))\, ds \right) - \exp\left(2 \int_0^t \tilde{\theta}_s^{T_{k+1}}(\lambda(s,T_l))\, ds \right).$$

Proof: For $l \in \{i, j\}$, we have

$$L_t^{T_{l+1}} = \tilde{L}_t^{T_{k+1}} + \int_0^t \left(b_s^{T_{k+1}} + d_s^{l,k} \right) ds.$$

Theorem 1 leads to

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[L(t,T_l)\right] \\
&\quad= L(0,T_l)\exp\left(\int_0^t \left\langle \lambda(s,T_l), d_s^{l,k} + b_s^{T_{k+1}} \right\rangle ds\right)\exp\left(\int_0^t \tilde{\theta}_s^{T_{k+1}}(\lambda(s,T_l))\, ds\right), \\
&\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[L(t,T_l)^2\right] \\
&\quad= L(0,T_l)^2\exp\left(2\int_0^t \left\langle \lambda(s,T_l), d_s^{l,k} + b_s^{T_{k+1}} \right\rangle ds\right)\exp\left(\int_0^t \tilde{\theta}_s^{T_{k+1}}(2\lambda(s,T_l))\, ds\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Var}_{\mathbb{P}_{T_{k+1}}}\left(L(t,T_l)\right) \\
&\quad= L(0,T_l)^2\exp\left(2\int_0^t \left\langle \lambda(s,T_l), d_s^{l,k} + b_s^{T_{k+1}} \right\rangle ds\right) \\
&\qquad\cdot\left(\exp\left(\int_0^t \tilde{\theta}_s^{T_{k+1}}\big(2\lambda(s,T_l)\big)\, ds\right) - \exp\left(2\int_0^t \tilde{\theta}_s^{T_{k+1}}(\lambda(s,T_l))\, ds\right)\right).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[L(t,T_i)L(t,T_j)\right] \\
&\quad= g_1(t,i,j,k)\, L(0,T_i)L(0,T_j) \\
&\qquad\cdot\exp\left(\int_0^t \left(\left\langle \lambda(s,T_i), d_s^{i,k} + b_s^{T_{k+1}} \right\rangle + \left\langle \lambda(s,T_j), d_s^{j,k} + b_s^{T_{k+1}} \right\rangle\right) ds\right)
\end{aligned}$$

and the proof is completed by

$$\begin{aligned}
&\mathrm{Corr}_{\mathbb{P}_{T_{k+1}}}(L(t,T_i), L(t,T_j)) \\
&\quad= \frac{\mathbb{E}_{\mathbb{P}_{T_{k+1}}}[L(t,T_i)L(t,T_j)] - \mathbb{E}_{\mathbb{P}_{T_{k+1}}}[L(t,T_i)]\mathbb{E}_{\mathbb{P}_{T_{k+1}}}[L(t,T_j)]}{\sqrt{\mathrm{Var}_{\mathbb{P}_{T_{k+1}}}(L(t,T_i))}\sqrt{\mathrm{Var}_{\mathbb{P}_{T_{k+1}}}(L(t,T_j))}}. \qquad \square
\end{aligned}$$

In the same way, the correlation of $L(t_1,T_i)$ and $L(t_2,T_j)$ can be calculated for different times t_1 and t_2. Let i, j, $k \in \{1,\ldots,N\}$ and $0 \le t_1 \le t_2 \le \min\{T_i, T_j\}$. Then, given the measure $\mathbb{P}_{T_{k+1}}$ (and under the approximation), the correlation of the LIBOR rates $L(t_1,T_i)$ and $L(t_2,T_j)$ is

$$\begin{aligned}
&\mathrm{Corr}_{\mathbb{P}_{T_{k+1}}}(L(t_1,T_i), L(t_2,T_j)) \\
&\quad= \exp\left(\int_{t_1}^{t_2} \tilde{\theta}_s^{T_{k+1}}(\lambda(s,T_j))\, ds\right)\frac{g_1(t_1,i,j,k) - g_2(t_1,i,j,k)}{\sqrt{h(t_1,i,k)}\sqrt{h(t_2,j,k)}},
\end{aligned}$$

where g_1, g_2 and h are defined as in Theorem 3.

A suitable volatility structure for the LIBOR rates is given by

$$\lambda(t,T_i) = a(T_i - t)\exp(-b(T_i - t)) + c \tag{31}$$

with parameters a, b and c (see [14]). Here, we set $a = 1$ as this parameter can be included in the driving process. The figures 3–8 show the correlation of LIBOR rates in the Lévy LIBOR model under the measure $\mathbb{P}_{T_{N+1}}$. We have assumed that $L^{T_{N+1}}$ is a Lévy process which is generated by a normal inverse Gaussian distribution (see section 3).

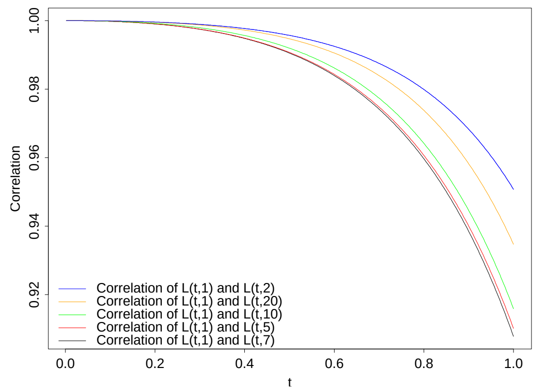

Figure 3: Correlations of LIBOR rates for $\alpha = 100$, $\beta = 0$, $\delta = 0.01$, $b = 0.5$ and $c = 0.1$

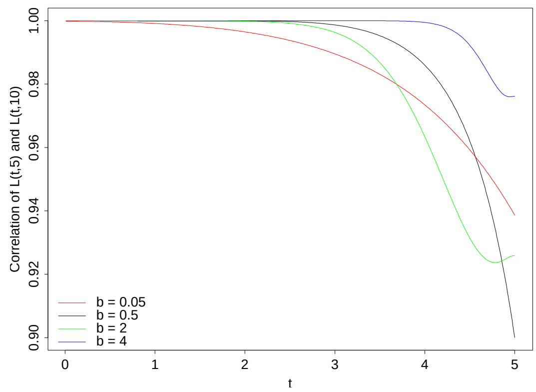

Figure 4: Correlation of $L(t, 5)$ and $L(t, 10)$ for $\alpha = 100$, $\beta = 0$, $\delta = 0.01$ and $c = 0.1$

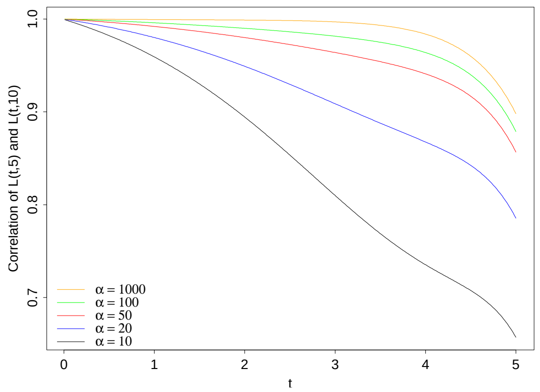

Figure 5: Correlation of $L(t, 5)$ and $L(t, 10)$ for $\beta = 0$, $\delta = 10$, $b = 0.5$ and $c = 0.1$

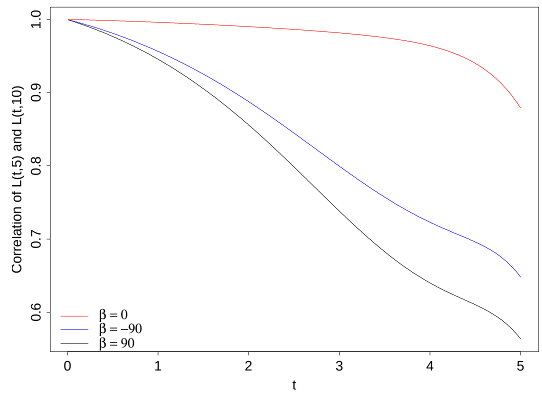

Figure 6: Correlation of $L(t, 5)$ and $L(t, 10)$ for $\alpha = 100$, $\delta = 10$, $b = 0.5$ and $c = 0.1$

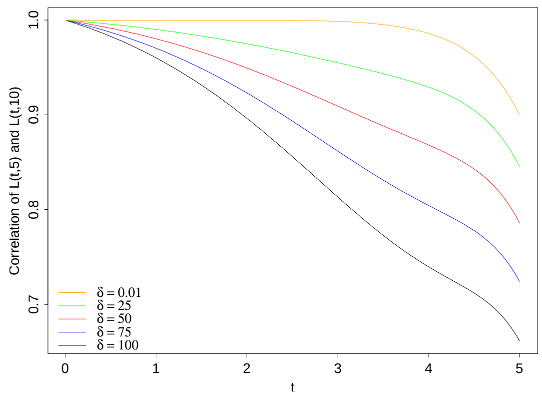

Figure 7: Correlation of $L(t, 5)$ and $L(t, 10)$ for $\alpha = 100$, $\beta = 0$, $b = 0.5$ and $c = 0.1$

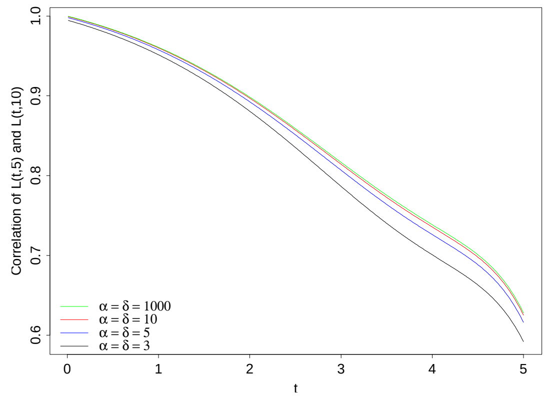

Figure 8: Correlation of $L(t, 5)$ and $L(t, 10)$ for $\beta = 100$, $b = 0.5$ and $c = 0.1$

5 The Lévy forward process model

In the Lévy forward process model the forward processes $F(\cdot, T_i, T_{i+1})$ are modeled by time-inhomogeneous Lévy processes in a similar way as the LIBOR rates have been modeled in the last section. The LIBOR rates can be deduced from the forward processes by

$$L(t,T) = \frac{1}{\delta}(F(t,T,T+\delta) - 1). \tag{32}$$

The Lévy forward process model is derived by a backward induction similar to the one of the Lévy LIBOR model (see [9] and [5]). This approach has the advantage that the driving processes remain time-inhomogeneous Lévy processes so that any approximation can be avoided.

Let $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = T^*$ denote a discrete tenor structure and set $\delta_k := T_{k+1} - T_k$. The construction of the model starts again with a d-dimensional time-inhomogeneous Lévy process $L^{T_{N+1}}$ on a complete stochastic basis $(\Omega, \mathcal{F}_{T_{N+1}}, \mathbb{F}, \mathbb{P}_{T_{N+1}})$ which satisfies assumption ($\mathbb{EM}$). Its canonical representation is given by (15). We make the following assumptions.

Assumption ($\mathbb{FP}$.1)**:** *For every T_i there is a bounded, continuous, deterministic function $\lambda(\cdot, T_i) : [0, T^*] \to [-M/2, M/2]^d$ which represents the volatility of the forward process $F(\cdot, T_i, T_{i+1})$. We assume that for all $k \in \{1, \ldots, N\}$*

$$\left| \sum_{i=1}^{k} \lambda^j(s, T_i) \right| \leq M$$

holds for all $s \in [0, T^*]$ and all $j \in \{1, \ldots, d\}$. Here, M is the constant from assumption ($\mathbb{EM}$) *and we set $\lambda(s, T_i) = 0$ for $s > T_i$.*

Assumption ($\mathbb{FP}$.2)**:** *The initial values of the zero coupon bond prices $B(0, T_i)$, $i \in \{1, \ldots, N+1\}$, are strictly positive. Consequently, the initial values of the forward processes are given by $F(0, T_i, T_{i+1}) = B(0, T_i)/B(0, T_{i+1})$.*

Then the forward processes $F(\cdot, T_i, T_{i+1})$ are given by

$$F(t, T_i, T_{i+1}) := F(0, T_i, T_{i+1}) \exp\left(\int_0^t \lambda(s, T_i)\, dL_s^{T_{i+1}} \right) \tag{33}$$

where $L^{T_{i+1}}$ are time-inhomogeneous Lévy processes. For $i \in \{1, \ldots, N\}$, the driving processes $L^{T_{i+1}}$ and the forward martingale measures $\mathbb{P}_{T_{i+1}}$ are constructed by a backward induction in a similar way as in section 4.1. Here, the time-inhomogeneous Lévy processes $L^{T_{i+1}}$ differ only by deterministic drift terms. Define $\widetilde{\theta}_s^{T_{i+1}}$ as in (30).

The correlations of the forward processes and the LIBOR rates in the Lévy forward process model can now be calculated in exactly the same way as in the Lévy LIBOR model in section 4.2.

Let $i, j, k \in \{1, \ldots, N\}$ and $0 \leq t \leq \min\{T_i, T_j\}$. Then under the measure $\mathbb{P}_{T_{k+1}}$ the correlation of $F(t, T_i, T_{i+1})$ and $F(t, T_j, T_{j+1})$ is given by

$$\mathrm{Corr}_{\mathbb{P}_{T_{k+1}}}(F(t, T_i, T_{i+1}), F(t, T_j, T_{j+1})) = \frac{g_1(t,i,j,k) - g_2(t,i,j,k)}{\sqrt{h(t,i,k)}\sqrt{h(t,j,k)}}, \tag{34}$$

where

$$g_1(t,i,j,k)) := \exp\left(\int_0^t \tilde{\theta}_s^{T_{k+1}}\Big(\lambda(s,T_i)+\lambda(s,T_j)\Big)\,ds\right),$$

$$g_2(t,i,j,k)) := \exp\left(\int_0^t \left(\tilde{\theta}_s^{T_{k+1}}\big(\lambda(s,T_i)\big)+\tilde{\theta}_s^{T_{k+1}}\big(\lambda(s,T_j)\big)\right)\,ds\right)$$

and for $l \in \{i,j\}$ we set

$$h(t,l,k) := \exp\left(\int_0^t \tilde{\theta}_s^{T_{k+1}}\big(2\lambda(s,T_l)\big)\,ds\right) - \exp\left(2\int_0^t \tilde{\theta}_s^{T_{k+1}}\big(\lambda(s,T_l)\big)\,ds\right).$$

Under $\mathbb{P}_{T_{k+1}}$, the LIBOR rates $L(t,T_i)$ and $L(t,T_j)$ have the same correlation as $F(t,T_i,T_{i+1})$ and $F(t,T_j,T_{j+1})$. This follows immediately from (32).

Let $0 \le t_1 \le t_2 \le \min\{T_i,T_j\}$. The correlation of $F(t_1,T_i,T_{i+1})$ and $F(t_2,T_j,T_{j+1})$ under $\mathbb{P}_{T_{k+1}}$ is

$$\begin{aligned}&\mathrm{Corr}_{\mathbb{P}_{T_{k+1}}}(F(t_1,T_i,T_{i+1}),F(t_2,T_j,T_{j+1}))\\ &\qquad = \exp\left(\int_{t_1}^{t_2} \tilde{\theta}_s^{T_{k+1}}(\lambda(s,T_j))\,ds\right)\frac{g_1(t_1,i,j,k)-g_2(t_1,i,j,k)}{\sqrt{h(t_1,i,k)}\sqrt{h(t_2,j,k)}}\end{aligned} \tag{35}$$

with g_1, g_2 and h defined as above. The LIBOR rates $L(t_1,T_i)$ and $L(t_2,T_j)$ again have the same correlation as $F(t_1,T_i,T_{i+1})$ and $F(t_2,T_j,T_{j+1})$ under $\mathbb{P}_{T_{k+1}}$.

6 The cross-currency Lévy LIBOR model

The cross-currency Lévy LIBOR model is an extension of the Lévy LIBOR model to a multi-currency setting (see Eberlein and Koval [7]).

6.1 Construction of the model

We consider $m+1$ markets with different currencies. They are labelled with indices $0,1,\dots,m$ where 0 denotes the domestic market. Let $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = T^*$ be a discrete tenor structure which is assumed to be the same for every market. Set $\delta_k = T_{k+1} - T_k$. The construction of the model starts with a d-dimensional time-inhomogeneous Lévy process $L^{0,T_{N+1}}$ on a complete stochastic basis $(\Omega, \mathcal{F}_{T_{N+1}}, \mathbb{F}, \mathbb{P}^{0,T_{N+1}})$ which satisfies assumption $(\mathbb{EM})$. The canonical representation of $L^{0,T_{N+1}}$ is given by

$$L_t^{0,T_{N+1}} = \int_0^t b_s^{0,T_{N+1}}\,ds + \int_0^t \sqrt{c_s}\,dW_s^{0,T_{N+1}} + \int_0^t\int_{\mathbb{R}^d} x\,(\mu-\nu^{0,T_{N+1}})(ds,dx), \tag{36}$$

where $W^{0,T_{N+1}}$ is a d-dimensional standard Brownian motion under $\mathbb{P}_{0,T_{N+1}}$. μ is the random measure of jumps of $L^{0,T_{N+1}}$ and $\nu^{0,T_{N+1}}$ is the $\mathbb{P}_{0,T_{N+1}}$-compensator of μ.

As a first step, a model for the forward exchange rates is designed. It is based on the following assumptions.

Assumption $(\mathbb{FXR}.1)$: *For every market $i \in \{0,1,\dots,m\}$ there is a strictly positive family of zero coupon bond prices $B^i(0,T_j)$, $j \in \{1,\dots,N+1\}$, which is strictly decreasing in j. Furthermore, we assume that positive spot exchange rates $X^i(0)$ are* *given as initial values (expressed in units of the domestic currency per unit of the foreign currency). Then the initial value of the forward exchange rate for time T^* is given by*

$$F_{X^i}(0,T^*) = \frac{B^i(0,T^*)X^i(0)}{B^0(0,T^*)}.$$

Assumption ($\mathbb{FXR}$.2): *For every foreign market $i \in \{1,\dots,m\}$ there is a continuous deterministic function $\xi^i(\cdot,T^*) : [0,T^*] \to \mathbb{R}^d$. We assume that there is a constant $\overline{M} \in (0, M/(N+2))$ such that*

$$0 \le (\xi^i(s,T^*))_k \le \overline{M}$$

for all components $k \in \{1,\dots,d\}$, all $s \in [0,T^*]$ and all $i \in \{1,\dots,m\}$. Here, M is the constant from assumption ($\mathbb{EM}$).

Assumption ($\mathbb{FXR}$.3): *For every $i \in \{1,\dots,m\}$ the forward exchange rate for T^* is given by*

$$F_{X^i}(t,T^*) := F_{X^i}(0,T^*) \exp\left(\int_0^t \gamma^i(s,T^*)\,ds + \int_0^t \xi^i(s,T^*)\,dL_s^{0,T_{N+1}}\right),$$

where the drift coefficients $\gamma^i(\cdot,T^*)$ are chosen in such a way that $F_{X^i}(\cdot,T^*)$ is a martingale under $\mathbb{P}_{0,T_{N+1}}$.

For every foreign market $i \in \{1,\dots,m\}$, forward martingale measures $\mathbb{P}_{i,T_{N+1}}$ associated with $T_{N+1} = T^*$ are defined by the Radon–Nikodym derivative

$$\left.\frac{d\mathbb{P}_{i,T_{N+1}}}{d\mathbb{P}_{0,T_{N+1}}}\right|_{\mathcal{F}_t} = \frac{F_{X^i}(t,T^*)}{F_{X^i}(t,T^*)}. \tag{37}$$

According to Girsanov's theorem, $L^{0,T_{N+1}}$ is a time inhomogeneous Lévy process under $\mathbb{P}_{i,T_{N+1}}$ with the canonical representation

$$L_t^{0,T_{N+1}} = \int_0^t \tilde{b}_s^{i,T_{N+1}}\,ds + \int_0^t \sqrt{c_s}\,dW_s^{i,T_{N+1}} + \int_0^t \int_{\mathbb{R}^d} x\,(\mu - \nu^{i,T_{N+1}})(ds,dx), \tag{38}$$

where $W^{i,T_{N+1}}$ is a d-dimensional standard Brownian motion under $\mathbb{P}_{i,T_{N+1}}$ and $\nu^{i,T_{N+1}}$ is the $\mathbb{P}_{i,T_{N+1}}$-compensator of μ.

Then, we define driving processes $L^{i,T_{N+1}}$ for every foreign market $i \in \{1,\dots,d\}$ by

$$L_t^{i,T_{N+1}} := \int_0^t b_s^{i,T_{N+1}}\,ds + \int_0^t \sqrt{c_s}\,dW_s^{i,T_{N+1}} + \int_0^t \int_{\mathbb{R}^d} x\,(\mu - \nu^{i,T_{N+1}})(ds,dx). \tag{39}$$

Here, $W^{i,T_{N+1}}$ and $\nu^{i,T_{N+1}}$ are given by (38) and $b^{i,T_{N+1}}$ is specified during the construction of the LIBOR rates. The time-inhomogeneous Lévy processes $L^{i,T_{N+1}}$ differ only by deterministic drift terms.

Now, a model for the LIBOR rates $L^i(\cdot,T_j)$ in every market $i \in \{0,1,\dots,m\}$ can be constructed. In every market, i.e. keeping i fixed, the construction is exactly the same as the construction of the Lévy LIBOR model in section 4. We obtain driving processes $L^{i,T_{j+1}}$ and measures $\mathbb{P}_{i,T_{j+1}}$ so that the LIBOR rates $L^i(\cdot,T_j)$ are given in the form

$$L^i(t,T_j) = L^i(0,T_j) \exp\left(\int_0^t \lambda^i(s,T_j)\,dL_s^{i,T_{j+1}}\right). \tag{40}$$

As in section 4, the LIBOR rates $L^i(\cdot,T_j)$ are martingales under $\mathbb{P}_{i,T_{j+1}}$.

6.2 Correlations of the LIBOR rates

The correlations of the LIBOR rates in the cross-currency Lévy LIBOR model can be calculated in the same way as in the Lévy LIBOR model. Here, we use again the approximation which has been introduced in section 4.2.

With this approximation, all the driving processes $L^{i,T_{j+1}}$ are time-inhomogeneous Lévy processes which differ only by deterministic drift terms. $L^{i,T_{j+1}}$ has the canonical representation

$$L_t^{i,T_{j+1}} = \int_0^t b_s^{i,T_{j+1}}\, ds + \int_0^t \sqrt{c_s}\, dW_s^{i,T_{j+1}} + \int_0^t \int_{\mathbb{R}^d} x\,(\mu - \nu^{i,T_{j+1}})(ds,dx), \tag{41}$$

where $\nu^{i,T_{j+1}}(ds,dx) = F_s^{i,T_{j+1}}(dx)\, ds$ is the $\mathbb{P}_{i,T_{j+1}}$-compensator of μ. Define

$$\tilde{\theta}_s^{i,T_{j+1}}(z) := \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,x\rangle} - 1 - \langle z,x\rangle\right) F_s^{i,T_{j+1}}(dx). \tag{42}$$

Let $i_1, i_2, l \in \{0,1,\dots,m\}$ and $j_1, j_2, k \in \{1,\dots,N\}$. Consider $t \in [0,T^*]$ such that $0 \le t \le \min\{T_{j_1}, T_{j_2}\}$. Then the correlation of the LIBOR rates $L^{i_1}(t,T_{j_1})$ and $L^{i_2}(t,T_{j_2})$ under the measure $\mathbb{P}_{l,T_{k+1}}$ is

$$\mathrm{Corr}_{\mathbb{P}_{l,T_{k+1}}}(L^{i_1}(t,T_{j_1}), L^{i_2}(t,T_{j_2})) = \frac{g_1(t,i_1,i_2,j_1,j_2,l,k) - g_2(t,i_1,i_2,j_1,j_2,l,k)}{\sqrt{h(t,i_1,j_1,l,k))}\sqrt{h(t,i_2,j_2,l,k)}}, \tag{43}$$

where

$$g_1(t,i_1,i_2,j_1,j_2,l,k)) := \exp\left(\int_0^t \tilde{\theta}_s^{l,T_{k+1}}\left(\lambda^{i_1}(s,T_{j_1}) + \lambda^{i_2}(s,T_{j_2})\right) ds\right),$$
$$g_2(t,i_1,i_2,j_1,j_2,l,k)) := \exp\left(\int_0^t \left(\tilde{\theta}_s^{l,T_{k+1}}(\lambda^{i_1}(s,T_{j_1})) + \tilde{\theta}_s^{l,T_{k+1}}(\lambda^{i_2}(s,T_{j_2}))\right) ds\right)$$

and for $p \in \{1,2\}$ we define

$$h(t,i_p,j_p,l,k) := \exp\left(\int_0^t \tilde{\theta}_s^{l,T_{k+1}}(2\lambda^{i_p}(s,T_{j_p}))\, ds\right) - \exp\left(2\int_0^t \tilde{\theta}_s^{l,T_{k+1}}(\lambda^{i_p}(s,T_{j_p}))\, ds\right).$$

Now, consider different times t_1 and t_2 such that $0 \le t_1 \le t_2 \le \min\{T_{j_1}, T_{j_2}\}$. Then the correlation of the LIBOR rates $L^{i_1}(t_1,T_{j_1})$ and $L^{i_2}(t_2,T_{j_2})$ under the measure $\mathbb{P}_{l,T_{k+1}}$ is given by

$$\begin{aligned}&\mathrm{Corr}_{\mathbb{P}_{l,T_{k+1}}}(L^{i_1}(t_1,T_{j_1}), L^{i_2}(t_2,T_{j_2}))\\ &\quad = \exp\left(\int_{t_1}^{t_2} \tilde{\theta}_s^{l,T_{k+1}}(\lambda^{i_2}(s,T_{j_2}))\, ds\right) \frac{g_1(t_1,i_1,i_2,j_1,j_2,l,k) - g_2(t_1,i_1,i_2,j_1,j_2,l,k)}{\sqrt{h(t_1,i_1,j_1,l,k)}\sqrt{h(t_2,i_2,j_2,l,k)}},\end{aligned} \tag{44}$$

where g_1, g_2 and h are defined as above.

7 Calibration of the Lévy forward rate model

The correlation formulas which we derived in section 3.1 allow to calibrate the Lévy forward rate model based on empirical correlations. The calibration is done in two steps. In a first step, we estimate correlations between prices of zero coupon bonds using yield curve data. Then in a second step, we use the correlations to estimate the parameters of the driving Lévy process. In the implementation of the estimation procedure we use a Ho–Lee volatility structure

$$\Sigma(s,T) = \sigma_0(T-s). \tag{45}$$

Without loss of generality we set $\sigma_0 = 1$. As driving processes we consider NIG Lévy processes and Brownian motions. The special case of the Lévy forward rate model driven by a Brownian motion is the HJM model. The method described below can be used in the same way for other stationary volatility structures and general GH processes.

The underlying data set consists of yield curve estimates expressed in terms of their Svensson parameters ([15]). These parameters were estimated by the Deutsche Bundesbank on a daily basis using German government bonds that are listed at the Frankfurt exchange. We use sets of parameters derived from quotes starting August 7, 1997 and ending April 9, 2008, i.e. we consider 2707 trading days. Based on this data set we get $B(t,T)$ for every day t and any traded maturity T.

In order to estimate correlations, we need independent samples for each price. To create these samples we define the following quantities

Lemma 4 *For fixed t, T, Δ, where $t < T$ and $\Delta \in [0, T^* - T]$, define the random variable*

$$B^{\Delta}(t,T) := B(t+\Delta, T+\Delta)\frac{B(0,T)}{B(0,t)}\frac{B(\Delta,t+\Delta)}{B(\Delta,T+\Delta)}, \tag{46}$$

then $B^{\Delta}(t,T)$ has the same distribution as $B(t,T)$. Furthermore for $\Delta \geq t$, $B^{\Delta}(t,T)$ is independent of $B(t,T)$.

Proof: Using the representation (6) for $B(t+\Delta,T+\Delta)$, $B(\Delta,T+\Delta)$ and $B(\Delta,t+\Delta)$ we get

$$\begin{aligned} &B(t+\Delta,T+\Delta) \\ &= \frac{B(\Delta,T+\Delta)}{B(\Delta,t+\Delta)} \exp\left(-\int_{\Delta}^{t+\Delta} A(s,t+\Delta,T+\Delta)\,ds + \int_{\Delta}^{t+\Delta} \Sigma(s,t+\Delta,T+\Delta)\,dL_s\right). \end{aligned}$$

Inserting the Ho–Lee volatility structure and using the stationarity of L and Σ finishes the proof. Note that we also assume stationarity of the drift coefficient, i.e. $A(s,T) = A(0,T-s)$ for all $s \leq T$. $\square$

As a consequence, we get independent random variables $B^{\Delta}(t,T)$ if we choose $\Delta = t, 2t, 3t, \ldots$ Based on the given dataset, we calculate now the values of $B^{\Delta}(t,T)$ for any discrete t and T with $0 \leq t \leq T \leq T^*$ and $\Delta \in \{t, 2t, 3t, \ldots\}$. These will be used as samples to estimate $\mathrm{Corr}(B(t,T_1), B(t,T_2))$. Note that the samples have to satisfy the following equation

Lemma 5

$$\mathrm{Corr}(B^{\Delta}(t,T_1), B^{\Delta}(t,T_2)) = \mathrm{Corr}(B(t,T_1), B(t,T_2)) \qquad \forall\, \Delta \geq t.$$

Proof: Using the Ho–Lee volatility structure, we get

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\int_0^t \Sigma(s,t,T_1)\,d(L_{\Delta+s}-L_\Delta)+\int_0^t \Sigma(s,t,T_2)\,d(L_{\Delta+s}-L_\Delta)\right)\right]\\&\quad=\mathbb{E}[\exp((T_1+T_2-2t)(L_{\Delta+t}-L_\Delta))].\end{aligned}\tag{47}$$

With (47) and the stationarity of L we have

$$\mathbb{E}[B^\Delta(t,T_1)\,B^\Delta(t,T_2)]=\mathbb{E}[B(t,T_1)\,B(t,T_2)],$$
$$\mathbb{E}[B^\Delta(t,T_i)]=\mathbb{E}[B(t,T_i)]\quad\text{and}\quad\mathrm{Var}(B^\Delta(t,T_i))=\mathrm{Var}(B(t,T_i))\ \text{for } i\in\{1,2\}.\ \square$$

As an estimator for $\mathrm{Corr}(B(t,T_1),B(t,T_2))$ we use the usual empirical correlation given by

$$\begin{aligned}&\hat{\mathrm{C}}\mathrm{orr}(B(t,T_1),B(t,T_2))\\&\quad=\frac{\sum_{i=0}^n(B^{it}(t,T_1)-\bar{B}(t,T_1))(B^{it}(t,T_2)-\bar{B}(t,T_2))}{\sqrt{\sum_{i=0}^n(B^{it}(t,T_1)-\bar{B}(t,T_1))^2}\sqrt{\sum_{i=0}^n(B^{it}(t,T_2)-\bar{B}(t,T_2))^2}},\end{aligned}\tag{48}$$

whereas $n=[2707/t]$. $\bar{B}(t,T_1)$ and $\bar{B}(t,T_2)$ are the arithmetic means of $B^{0t}(t,T_1),\dots,B^{nt}(t,T_1)$ and $B^{0t}(t,T_2),\dots,B^{nt}(t,T_2)$ respectively. Evidently the smaller the value of t is chosen, the more samples can be used for the estimation.

In a second step, we use the estimated correlations to estimate the parameters α, β and δ of the NIG distribution and the parameter σ of the normal distribution. As mentioned in section 3.2, correlations do not depend on the parameter μ of the NIG distribution. Consequently there is no possibility and also no need to estimate μ. The same holds for the parameter μ of the normal distribution.

Note that for Lévy processes θ_s in (4) does not depend on s. Therefore using the relation $\theta(u)=\log(\Phi(-iu))$ and equation (13), one can write θ as a function of the parameters of the NIG distribution. For the Brownian motion, we get $\theta(u)=\mu u+\frac{1}{2}\sigma^2u^2$ using equation (4). Together with Theorem 2 correlations can be expressed in terms of the parameters of the driving process.

Now we estimate the parameters using the method of least squares. As an estimator $(\hat{\alpha},\hat{\beta},\hat{\delta})$ for (α,β,δ) we use the parameters which minimize the following function

$$\sum_{t=1\text{ day}}^{100\text{ days}}\ \sum_{T_1=1\text{ year}}^{10\text{ years}}\ \sum_{T_2=1\text{ year}}^{10\text{ years}}(\hat{\mathrm{C}}\mathrm{orr}(B(t,T_1),B(t,T_2))-\mathrm{Corr}(B(t,T_1),B(t,T_2)))^2.\tag{49}$$

The same objective function is used to get the parameter σ in case of the normal distribution. A maximum of 100 days is used for t. For greater values of t, the estimation of correlations would become too instable, as explained above. For T_1 and T_2 we use 1 to 10 years, because this is the time period which underlies the estimation of the Svensson parameters.

Table 1 shows the estimated parameter values. Figure 9 shows the estimated correlations as points. The lines represent the correlations which are calculated from the formula in Theorem 2 using the parameters given in Table 1. The figure shows that using general Lévy processes, one gets a good fit, while using Brownian motions, the model cannot produce realistic correlations.

Table 1: Estimated parameters

NIG Lévy process: $\hat{\alpha}$= 2851.521, $\hat{\beta}$= -2841.248, $\hat{\delta}$= 4.36e-15	
Brownian motion:	$\hat{\sigma}$= 5.713723e-05

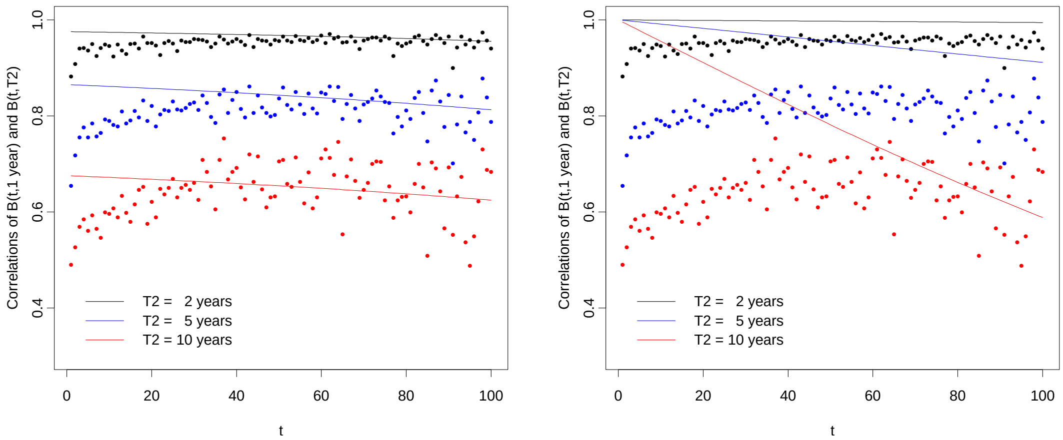

Figure 9: Empirical correlations (points) and correlations calculated from the models (lines) for the calibrations with NIG Lévy processes (left) and with Brownian motions (right)

Conclusion

We have derived explicit formulae for the correlations of interest rates for different maturities in Lévy interest rate models. The full range of forward rate (HJM), LIBOR and forward process models driven by Lévy or more general processes has been considered. We have also discussed the extension to a multicurrency setting.

The numerical implementation shows that in all models the correlations increase if the driving process approaches a Brownian motion. This is a consequence of the restricted flexibility of Brownian motion compared to other Lévy processes. In the forward rate model the correlations of zero coupon bond prices decrease monotonically if the distance of the maturities grows. This behaviour cannot be observed for correlations in the LIBOR model. Here the correlations decline at first but then start to increase again. Finally we calibrate the Lévy forward rate model using empirical correlations from German government bond price data. The failure to model correlations with a Brownian motion driven approach becomes evident.

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