Nested Interpolants

Matthias Heizmann  Jochen Hoenicke  Andreas Podelski
University of Freiburg, Germany

Abstract
In this paper, we explore the potential of the theory of nested words for partial correctness proofs of recursive programs. Our conceptual contribution is a simple framework that allows us to shine a new light on classical concepts such as Floyd/Hoare proofs and predicate abstraction in the context of recursive programs. Our technical contribution is an interpolant-based software model checking method for recursive programs. The method avoids the costly construction of the abstract transformer by constructing a nested word automaton from an inductive sequence of ‘nested interpolants’ (i.e., interpolants for a nested word which represents an infeasible error trace).

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1. Introduction
A recent trend in software verification tools in the line of [5, 8, 9, 12, 14, 15, 16, 19, 20] is to use interpolants in order to avoid the construction of the abstract transformer for the underlying program analysis. The idea is to generate an inductive sequence of interpolants from a false positive (a ‘spurious error trace’) returned by the program analysis. Roughly, the inductiveness expresses that the interpolants can be used to form a sequence of Hoare triples which proves the infeasibility of the spurious error trace. The inductiveness thus entails the elimination of the false positive. While the approach is incarnated with great success in the ‘lazy abstraction with interpolants’ scheme for non-procedural programs [19], it is not clear how one can extend it to a principled method for the general class of programs with procedures, possibly recursive ones. What we are interested in here is a general foundation that allows us to systematically reason about interpolant-based correctness proofs for recursive programs. In this paper, we propose to base such a foundation on the theory of nested words [2, 3].

The motivation to use nested words stems directly from the observation that the stack-based semantics of recursive programs is not appropriate for our purpose. If we model a state as a stack of valuations of program variables and generate interpolants from a trace modeled as a sequence of states, then the interpolants cannot be restricted to contain variables of the local calling context. This defeats the very purpose of interpolants (to carry a minimal amount of information needed locally at a given position in the trace).

The theory of nested words (and nested word automata) offers an interesting potential as an alternative to the low-level view of a recursive program as a stack-based device that defines a set of traces. A nested word expresses not only the linear order of a trace but also the nesting of calls and returns. A nested word automaton does not need a stack. If we model the execution of a recursive program as a nested word, we can reason directly on the nesting, instead of having to recover the nesting with the help of a stack. We thus trade an operationally complex device defining a set of simple objects (traces) for an operationally simple device defining a set of rich objects (nested traces). The insight put forward in this work is that the increase of the object complexity is irrelevant since we reduce the reasoning of program correctness to the emptiness of the defined set of objects (the objects are existentially quantified and thus ‘projected out’).

In this paper, we explore the potential of the theory of nested words as a foundation for correctness proofs for the general class of recursive procedures. Our conceptual contribution is a simple framework that allows us to shine a new light on classical concepts such as Floyd/Hoare proofs and predicate abstraction for recursive programs. Our technical contribution is to give, to our knowledge for the first time, a principled method that constructs an abstract proof for recursive programs from interpolants (avoiding the construction of the abstract transformer). The method constructs a nested word automaton from an inductive sequence of ‘nested interpolants’ (i.e., interpolants for the ‘nested trace’ of the recursive program).
procedure m(x) returns (res)
ℓ₀: if x>100
    {x ≥ 101}
    ℓ₁: res:=x-10
else
    {x ≤ 100}
ℓ₂: xₘ := x+11
    {xₘ ≤ 111}
ℓ₃: call m
ℓ₄: xₘ := resₘ
    {resₘ ≤ 101}
ℓ₅: call m
    {resₘ = 91}
ℓ₆: res := resₘ
    {res = 91 ∨ (x ≥ 101 ∧ res = x - 10)}
ℓ₇: assert (x≥101 -> res=91)
return m

Figure 1. McCarthy’s 91 function together with a Floyd-Hoare style annotation with invariant assertions. The program is correct if the assert statement never fails.

inductive sequence of interpolants from the spurious error trace of a non-recursive (non-procedural) program. The interpolants form the labels in the abstract reachability tree (in conjunctions in [19], and in tuples in [13]). The method thus constructs an interpolant-based proof and avoids the construction of an abstract transformer.

The second method, as used, e.g., in [11, 14, 17], generates a not necessarily inductive sequence of interpolants from the spurious error trace of a recursive program. The interpolants are used solely for defining new predicates (by subformulas). The method does construct an abstract transformer. The proof is constructed by fixpoint iteration. The iterates (i.e., not the interpolants) form a sequence of Hoare triples which proves the infeasibility of the spurious error trace. I.e., the inductiveness of sequences of interpolants is irrelevant in [11, 14].

To summarize, the first method does not account for recursion, and the second one does not consider inductive sequences of interpolants (or other ways to avoid the construction of the abstract transformer). In contrast, we use nested words to formalize inductive sequences of interpolants for traces of recursive programs (and nested word automata to avoid the construction of the abstract transformer).

2. Preliminaries

In this section, we fix the notation for recursive programs (following [21]) and for nested words (following [2, 3]).

2.1 Recursive Programs

Informal Presentation. We consider a simple procedural programming language in a BoogiePL-like syntax [6]. As usual, we impose a number of restrictions and conventions in order to simplify the formal exposition. The language has no pointers, no global variables and only call-by-value procedure calls. It is forbidden to write to the input variables of a procedure. A procedure p has one input parameter x and one output parameter res. Procedure calls appear in the following form.

resₚ := call p(xₚ)

If procedure p with formal parameter x is called, the argument (actual parameter) is a special variable xₚ. On return, the value of the procedure’s output variable res is stored into the special variable resₚ of the caller. Before calling procedure p, the caller writes the input value to the variable xₚ. The callee accesses the value through the variable x. The callee stores the result value into the variable res. The result value is returned to the caller through the variable resₚ. Procedures with arbitrarily many input and output parameters can be treated in an analogous way. A global variable can be modeled by adding an input and an output variable to each procedure that directly or indirectly (via a recursive function call) accesses the global variable.

Figure 1 shows an implementation of McCarthy’s 91 function (if the argument x is not greater than 101, the function returns 91) in a code fragment that meets our restrictions and conventions.

**Formal Presentation: Recursive Control Flow Graph.** Following [21], we present a recursive program formally as a recursive control flow graph; see Figure 2 for an example. Each node is a program location ℓ. Each edge is labeled with a statement δ, which is either an assignment (δ:=), an assume (δφ), a call (call p), or a return (return p).

For each procedure p, the recursive control flow graph contains p’s control flow graph. The edges between internal nodes are labeled with assignment or assume statements. The nodes of p’s control flow graph are the program locations of p, say ℓ₀, . . . , ℓₙ, where the node ℓ₀ is the entry location of p and ℓₙ its exit location. To model that procedure p is called from procedure q at program location ℓₗ (in q’s control flow graph),

- an edge labeled (call p) goes from ℓₗ to ℓ₀, the entry location of p,
Figure 3. Semantics of statements in the context of a recursive control flow graph (where a statement is used to label an edge between two nodes \( \ell \) and \( \ell' \)). A state is a stack of local states/location-valuation pairs; the current local state is the topmost/rightmost one.

- an edge labeled \( \text{return } p \uparrow \ell' \) goes from \( \ell' \), the exit location of \( p \), to the location \( \ell_{j+1}' \) in \( q \)'s control flow graph, where \( \ell_{j+1}' \) is the successor location of \( \ell_j' \) with respect to the call of procedure \( p \).

In Figure 2, we use a thick gray edge in order to depict the successor location of a call location, with respect to the procedure calls initiated by \( \text{call } m \). The thick gray edges are not edges of the recursive control flow graph. The assert statement from Figure 1 is translated to an edge which is labeled with an assume statement and leads to a special error location.

Semantics. A valuation \( \nu \) is a function that maps program variables to values. A (local) state of a procedure is a pair \((\ell, \nu)\), consisting of a program location and a valuation. A state \( S \) of the (whole) program is a stack of local states, i.e., location-valuation pairs, which we write as a sequence

\[
S = (\ell_0, \nu_0), (\ell_1, \nu_1), \ldots, (\ell_n, \nu_n)
\]

Each element corresponds to a called (and not yet returned) procedure. The topmost/rightmost element represents the actual calling context.

Each statement \( \delta \) induces a transition relation between states; see Figure 3. We use the usual notation for a transition from a state \( S \) to a successor state \( S' \).

\[
S \xrightarrow{\delta} S'
\]

Traces \( \pi \). A trace \( \pi \) is a sequence of statements, \( \pi = \delta_0 \ldots \delta_{n-1} \).

We extend the transition relation from single statements to traces in the usual way and use the same notation also for traces.

\[
S \xrightarrow{\pi} S'
\]

A trace \( \pi \) is a feasible error trace of the program \( \mathcal{P} \) if there exists an initial valuation \( \nu_0 \), a stack of local states \( S \) and a final valuation \( \nu_n \) such that the triple

\[
(\ell_0, \nu_0) \xrightarrow{\pi} (\ell_{\text{err}}, \nu_n)
\]

is a transition of the program (where \( \ell_0 \) is the initial location of the main procedure of \( \mathcal{P} \) and \( \ell_{\text{err}} \) is an error location of \( \mathcal{P} \)).

Program Correctness = No Feasible Error Traces. The recursive control flow graph encodes program correctness through special nodes corresponding to error locations. We define the correctness of the program through the non-reachability of an error location \( \ell_{\text{err}} \) by a program execution. Formally, a program \( \mathcal{P} \) is correct if it has no feasible error trace.

The classical notion of partial correctness can be accommodated by representing precondition and postcondition by assume statements and assert statements (the latter is translated to an edge that leads to the error location).

2.2 Nested Words

Following [2, 3], a nested word over an alphabet \( \Sigma \) is a pair \((w, \rightsquigarrow)\) consisting of a word \( w = a_0 \ldots a_{n-1} \) over the alphabet \( \Sigma \) and the nesting relation \( \rightsquigarrow \). The fact that \( \rightsquigarrow \) is a nesting relation for \( w \) means that we have a relation between the \( n \) positions of \( w \), formally

\[
\rightsquigarrow \subseteq \{0, \ldots, n-1\} \times \{0, \ldots, n-1\} \times \{\infty\}
\]

which is left-unique, right-unique, and properly nested, formally:

\[
i_1 \rightsquigarrow j, \ i_2 \rightsquigarrow j, \ j \neq \infty \text{ implies } i_1 = i_2
\]

\[
i \rightsquigarrow j_1, \ i \rightsquigarrow j_2 \text{ implies } j_1 = j_2
\]

\[
i_1 \rightsquigarrow j_1, \ i_2 \rightsquigarrow j_2, \ i_1 \leq i_2 \text{ implies } \begin{cases} i_1 < j_1 < i_2 < j_2 \\ i_1 \leq i_2 < j_2 \leq j_1. \end{cases}
\]

The idea is that the relation between the position \( i \) of a call and the position \( j \) of a matching return is expressed by \( i \rightsquigarrow j \). Positions appearing on the left (resp. right) in pairs in \( \rightsquigarrow \) are called call (resp. return positions). All other positions are internal positions. The index \( \infty \) is used as return position for all unfinished calls. For a return position \( j \neq \infty \), the corresponding call position (i.e., the unique position \( i \) such that \( i \rightsquigarrow j \)) is the call predecessor of \( j \). In contrast with [3], we do not allow \( \rightsquigarrow \) as a call predecessor. In contrast with [2], we do allow \( \rightsquigarrow \) as a return position.

Possibly Infinite Nested Word Automata (NWA). A nestd word automaton over an alphabet \( \Sigma \) is a tuple

\[
\mathcal{A} = (Q, \delta_{\text{fin}}, \delta_{\text{ca}}, \delta_{\text{re}}, Q_{\text{fin}}, Q_{\text{fin}}^\text{out})
\]

consisting of

- a (not necessarily finite) set of states \( Q \),
- a triple \( \langle \delta_{\text{fin}}, \delta_{\text{ca}}, \delta_{\text{re}} \rangle \) of transition relations for, respectively, internal, call, and return positions,
- \( \delta_{\text{fin}} \subseteq Q \times \Sigma \times Q \),
- \( \delta_{\text{ca}} \subseteq Q \times Q \times Q \),
- \( \delta_{\text{re}} \subseteq Q \times Q \times \Sigma \times Q \),

- a set of initial states \( Q_{\text{fin}} \subseteq Q \),
- a set of final state \( Q_{\text{fin}}^\text{out} \subseteq Q \).

A run of a nested word automaton \( \mathcal{A} \) over the nested word \((q_0, \ldots, q_{n-1}, \rightsquigarrow)\) is a sequence \( q_0, \ldots, q_n \) of states that starts in an initial state, i.e., \( q_0 \in Q_{\text{fin}}^\text{out} \), and that is consecutive, i.e., for each \( i = 0, \ldots, n-1 \),

\[
(q_i, a_i, q_{i+1}) \in \delta_{\text{fin}} \text{ if } i \text{ is an internal position},
\]

\[
(q_i, a_i, q_{i+1}) \in \delta_{\text{ca}} \text{ if } i \text{ is a call position},
\]

\[
(q_i, a_i, q_{i+1}) \in \delta_{\text{re}} \text{ if } i \text{ is a return position and } k \rightsquigarrow i.
\]

The run is accepting if it ends in a final state, i.e., \( q_n \in Q_{\text{fin}}^\text{out} \).

The nested word automaton \( \mathcal{A} \) accepts the nested word \((w, \rightsquigarrow)\) if it has an accepting run over \((w, \rightsquigarrow)\). The language of nested words recognized by \( \mathcal{A} \) is the set \( L(\mathcal{A}) \) consisting of the nested words accepted by \( \mathcal{A} \).
 DeViating from [2, 3], we do not require that the set of states of a nested word automaton is finite. In the next section, we will define the data automaton as a possibly infinite nested word automaton. All other automata considered in this presentation are finite.

The notion of regularity does not deviate from [2, 3]. A language of nested words is regular if it is recognized by a finite nested word automaton. Regular languages of nested words enjoy the standard properties of regular language theory, of which we will use the closure under intersection and complement, and the decidability of emptiness [2, 3].

3. An NWA View of Program Correctness

The first step to formulate a view of program correctness in terms of nested words is to fix the alphabet, namely as the set of statements. Once we have done that it is very natural to give a view of Program correctness, to define a new proof rule based on nested word automata, and to give a new formulation of predicate abstraction based on nested word automata.

3.1 An NWA characterization of Program Correctness

We assume a fixed set of statements which we note \( \Sigma \). We will view \( \Sigma \) as an alphabet, statements \( \sigma \) as letters and traces \( \sigma_0 \ldots \sigma_{n-1} \) as words; i.e., \( \sigma_0 \ldots \sigma_{n-1} \in \Sigma^* \).

We define a nested trace to be a nested word
\[
\pi = (\sigma_0 \ldots \sigma_{n-1}, \rightsquigarrow)
\]
over the alphabet \( \Sigma \) of statements.

Given a nested trace over the alphabet \( \Sigma \) whose letters include call and return statements, it makes sense to refine the notion of nesting. We say that \( \pi = (\sigma_0 \ldots \sigma_{n-1}, \rightsquigarrow) \) is well-nested if the nesting relation between its positions is consistent with the letters at the respective positions; i.e., we have

- the letter \( \sigma_i \) is a call statement if and only if \( i \) is a call position, and
- the letter \( \sigma_i \) is a return statement if and only if \( i \) is a return position.

Example 1. Figure 4 depicts three examples of nested traces (over the alphabet of statements used in the McCarthy 91 function). The three corresponding nesting relations are given below.

\[
\begin{align*}
1 \rightsquigarrow & 4 \\
2 \rightsquigarrow & 5 \\
3 \rightsquigarrow & \infty
\end{align*}
\]

In the figure we use indentation to indicate the nesting relation.

Nested Traces and Possible Program Executions. The notion of a nested trace does not refer to a specific program (in fact, it may not correspond to a path in the given program’s recursive control flow graph), and it does not take into account the semantics of statements (it may not correspond to a possible execution of any program).

We will next introduce two properties of a nested trace which together imply the fact that it corresponds to a possible execution of the given program. The two properties characterize two orthogonal aspects (“control” vs. “data”) of this fact. Each of the two properties will be presented as a nested word automaton, \( A_P \) resp. \( A_\Sigma \).

The first property characterizes the fact that the nested trace complies with the control flow prescribed by the recursive program graph (and that it is well-nested). This property is a regular language of nested words; we will use the control automaton \( A_P \), a finite nested word automaton, to present it.

The second property characterizes that the nested trace complies with the semantics of statements as manipulators of data (and that it is well-nested). This property is in general not a regular language of nested words; we will use the data automaton \( A_\Sigma \), an in general infinite nested word automaton, to present it.

Control. From now on, we assume a fixed program \( P \) given formally as a recursive control flow graph. The set of nested traces that comply with the control expressed by \( P \) (ignoring the data) can be formally defined by the recursive program graph viewed as a nested word automaton. I.e., the edges of the recursive program graph defines the three kinds of transition relations; the initial location defines the set of initial states; the error location defines the set of final states.
Definition 1 (Control Automaton $A_P$, Nested Error Trace). Given the program $P$, the control automaton $A_P$ is the nested word automaton

$$A_P = (Q, \langle \delta_{in}, \delta_a, \delta_{re} \rangle, Q^{init}, Q^{fin})$$

where

- the set of states $Q$ is the set of nodes (i.e., program locations),
- the three transition relations $\delta_{in}, \delta_a, \delta_{re}$ are the edge relations; i.e., if the edge $(\ell, \ell')$ is labeled with:
  - the assignment or assume statement $s$, then
    $$(\ell, s, \ell') \in \delta_{in}$$
  - the call statement $\text{call } p$, then
    $$(\ell, \text{call } p, \ell') \in \delta_a$$
  - the return statement (with call location $\ell_c$, return $p$) $\text{return } p \uparrow \ell_c$, then
    $$(\ell, \ell_c, \text{return } p, \ell') \in \delta_{re}$$
- the set of initial states $Q^{init}$ consists of the initial location of the main procedure,
- the set of final states $Q^{fin}$ consists of the error location.

A nested error trace is a nested word accepted by the control automaton $A_P$.

An alternative, richer way to define the regular language of nested error traces uses the automata-theoretic product of two nested word automata. The first one is the recursive program graph (without error locations) viewed as a nested word automaton (just as we did for $A_P$) where every program location is a final state; i.e., $Q^{fin} = Q$. The second one is the nested word automata that corresponds to the negation of the CaRet property that defines the program correctness [1].

Example 2. The control automaton $A_{P_{91}}$ of the program $P_{91}$ is given in Figure 2. In the figure, a return edge labeled by a return statement together with the call location represents a transition in $\delta_a$. For example, the edge labeled $(\ell_7, \text{return } p, \ell_5, \ell_6)$ represents the transition $(\ell_7, \ell_5, \text{return } p, \ell_6)$ in $\delta_a$. We next take the three nested traces from Example 1 (given in Figure 4) and investigate whether they are accepted by $A_{P_{91}}$.

The nested trace $\tau_1$ is not accepted because position 1 is a call position according to the nesting relation (and hence the successor state should be given by the transition relation $\delta_a$), but the letter at position 1 is not a call statement (and hence there is no successor state according to the transition relation $\delta_a$).

The nested trace $\tau_2$ is not accepted. The run

$$r = \ell_0, \ell_2, \ell_3, \ell_0, \ell_1, \ell_7, \ell_6, \ell_7, \ell_{err}$$

is possible according to the graph structure of $A_{P_{91}}$. However, Position 5 of $\tau_2$ is a return position, and position 2 its call predecessor. Only if $(\ell_7, \ell_3, \text{return } p, \ell_0)$ was a transition in $\delta_a$ (corresponding to a return edge of $A_{P_{91}}$), $r$ would be a valid run.

This (the fact that the nested trace $\tau_2$ is not accepted) illustrates that the control automaton enforces the call/return discipline; i.e., return statements leading to $\ell_6$ can only return from procedures called from location $\ell_5$.

The nested trace $\tau_3$ is accepted by $A_{P_{91}}$. This illustrates the fact that the control automaton accepts traces with unmatched call statements. The example also illustrates that the existence of a nested error trace (i.e., the non-emptiness of the control automaton $A_P$) does not yet imply a violation of the correctness specification (“control is not enough; we also need data”).

Data. We next define the data automaton $A_S$. This (in general infinite) nested word automaton recognizes an (in general non-regular) set of nested traces. This set contains the well-nested traces that are possible according to the semantics of statements (whose execution manipulates data). We call these traces feasible nested traces.

This is the only point in the presentation where we define an infinite nested word automaton. In fact, from now on we will construct finite nested word automata that approximate the infinite data automaton $A_S$ (i.e., recognize a superset).

Definition 2 (Data Automaton $A_S$, Feasible Nested Trace). Given the set of states $\Sigma$, the data automaton

$$A_S = (Q, \langle \delta_{in}, \delta_a, \delta_{re} \rangle, Q^{init}, Q^{fin})$$

is the (in general infinite) nested word automaton where

- the set of states $Q$ is the (in general infinite) set of valuations $\nu$,
- the three transition relations $\delta_{in}, \delta_a, \delta_{re}$ are the transition relations induced by the statements in $\Sigma$; i.e., if the statement is:
  - the assignment statement $y := t$, then
    $$(\nu, y := t, \nu \oplus \{y \mapsto \nu(t)\}) \in \delta_{in}$$
  - the call statement $\text{call } p$, then
    $$(\nu, \text{call } p, \nu') \in \delta_a$$
  - the return statement $\text{return } p$, then
    $$(\nu, \text{return } p, \nu') \in \delta_{re}$$
- every state is an initial state (i.e., $Q^{init} = Q$),
- every state is a final state (i.e., $Q^{fin} = Q$).

A feasible nested trace is a nested word accepted by the data automaton.

Example 3. We take three nested traces from Example 1 (given in Figure 4) and investigate whether they are accepted by $A_S$. The nested trace $\tau_1$ is not accepted; the same explanation as with $A_P$ applies. In fact, both a control automaton and a data automaton accept only well-nested traces.

The nested trace $\tau_2$ is accepted (i.e., it is a feasible nested trace). The sequence $v_0, \ldots, v_n$ is an accepting run.

The nested trace $\tau_3$ is not accepted because there is no valuation/state $\nu$ with a run that reads the three letters: the assignment $x := 100$, the assignment $\text{res := x - 10}$ and the assignment $x \leftarrow x - 10$.

Theorem 1 (Characterizing Program Correctness by NWAs’s). The program $P$ is correct if and only if the intersection between the control automaton and the data automaton is empty, formally

$$\mathcal{L}(A_P) \cap \mathcal{L}(A_S) = \emptyset.$$ 

Proof. Using the three notions of feasible error trace (defined in Section 2.1), nested error trace (i.e., accepted by $A_P$), and feasible nested trace (i.e., accepted by $A_S$), we can rephrase the theorem as follows. A trace $s_0 \ldots s_{n-1}$ is a feasible error trace of $P$ if and only if there exists a nesting relation $\prec$ such that the nested trace $(s_0 \ldots s_{n-1}, \neg \prec)$ is at the same time a nested error trace and a feasible nested trace.
We show a more general statement by induction on the length \( n \) of the trace. For every nesting relation \( \leadsto \) such that \( \leadsto \subseteq \{0, \ldots, n-1\} \times \{0, \ldots, n-1, \infty\} \), every sequence of locations \( \ell_0, \ldots, \ell_n \) and every sequence of valuations \( \nu_0, \ldots, \nu_n \) the following two statements are equivalent.

- The sequence of locations \( \ell_0, \ldots, \ell_n \) is a run of \( A_P \) for \( (s_0 \ldots s_{n-1}) \leadsto \) and the sequence of valuations \( \nu_0, \ldots, \nu_n \) is a run of \( A_{\mathcal{V}} \) for \( (s_0 \ldots s_{n-1}) \leadsto \).

- The nested trace \( (s_0 \ldots s_{n-1}) \leadsto \) is well-nested and there is a sequence of stacks
  \[(\ell_0, \nu_0) \xrightarrow{a_1} S_1, (\ell_1, \nu_1) \xrightarrow{a_2} \ldots \xrightarrow{a_{n-1}} S_n, (\ell_n, \nu_n)\]
according to the transition relation of \( \mathcal{P} \). In that case \( S_1 = (\ell_{0,0} \nu_{0,0}) \ldots (\ell_{m,n} \nu_{m,n}) \) such that the nesting relation contains \( k_0 \leadsto k_1^0 \ldots k_m \leadsto k_m^0 \) and \( k_0 < \cdots < k_m < k_0^0 \leq \cdots \leq k_0^m \) holds.

### 3.2 Proof Rule Based on Finite NWA

The following proof rule for the correctness of recursive programs uses a finite nested word automaton \( \mathcal{A} \) as a proof argument: if \( \mathcal{A} \) recognizes a superset of the set of feasible nested traces and it does not intersect with the control automaton, then the program is correct.

\[ L(\mathcal{A}) \supseteq L(\mathcal{A}_S), \quad L(\mathcal{A}) \cap L(\mathcal{A}_P) = \emptyset \implies \mathcal{P} \text{ is correct} \quad (1) \]

The proof rule is interesting by the theory of regular languages of bitvectors. Completeness notoriously holds (if it holds) by a disappointingly trivial line of reasoning with no indication on how to find the proof argument. For the proof rule (1), it is sufficient to take the complement of \( \mathcal{A}_P \) as a candidate for \( \mathcal{A} \). Clearly its intersection with \( \mathcal{A}_P \) is empty. Since \( \mathcal{A}_P \) does not accept any feasible nested trace (otherwise \( \mathcal{P} \) would not be correct), or: it recognizes a subset of infeasible nested traces, its complement recognizes a superset of feasible traces. We summarize our discussion by the following statement.

**Theorem 2** (Soundness and Completeness of Proof Rule (1)). The program \( \mathcal{P} \) is correct if and only if there exists a finite nested word automaton \( \mathcal{A} \) that recognizes a superset of the set of feasible nested traces (i.e., \( L(\mathcal{A}) \supseteq L(\mathcal{A}_S) \)) and that does not intersect with the control automaton (i.e., \( L(\mathcal{A}) \cap L(\mathcal{A}_P) = \emptyset \)).

In the remainder of this presentation, we will investigate systematic ways to construct candidate proof arguments for a finite nested word automaton \( \mathcal{A} \).

### 3.3 Predicate Abstraction

In this section, we automate the proof rule given in the previous section by giving a predicate abstraction-based proof method for the correctness of a recursive program. The proof method uses predicate abstraction to construct a finite nested word automaton \( \mathcal{A}_{\text{Pred}} \) which recognizes a superset of the set of feasible nested traces. The proof succeeds if the intersection of \( \mathcal{A}_{\text{Pred}} \) with the control automaton \( \mathcal{A}_P \) is empty.

Intuitively, the proof method constructs \( \mathcal{A}_{\text{Pred}} \) by abstracting the data automaton \( \mathcal{A}_S \). The abstraction consists essentially of transforming the triple of transition relations between concrete valuations (the states of \( \mathcal{A}_S \)) into a triple of transition relations between abstract valuations. Abstract valuations are defined in terms of predicates; we will formally introduce abstract valuations as bitvectors.

**Predicates.** In our context, a predicate is a set of valuations. Such a set may be defined by an assertion, e.g., \( \pi \geq 101 \).

**Bitvectors.** Given a finite set of predicates, say \( \text{Pred} = \{p_1, \ldots, p_m\} \)
we call an \( m \)-tuple \( b \in \{0, 1\}^m \) a bitvector. Assuming a fixed order on the predicates, a bitvector \( b = (b_1, \ldots, b_m) \) has a meaning defined by

\[ \{(b_1, \ldots, b_m)\} = \{\nu \mid \forall j \in \{1, \ldots, m\}, \nu \in p_j \iff b_j = 1\}. \]

**Nested Post Operators** \( \langle \text{post}_{\text{int}}, \text{post}_{\text{ca}}, \text{post}_{\text{re}} \rangle \). The triple of nested post operators \( \langle \text{post}_{\text{int}}, \text{post}_{\text{ca}}, \text{post}_{\text{re}} \rangle \) defined in Figure 5 are in direct connection with the triple of transition relations \( \langle \delta_0, \delta_{ca}, \delta_{re} \rangle \) of the data automaton \( \mathcal{A}_S \). Namely, for a set \( \mathcal{V} \) of valuations, we have:
\[
\begin{align*}
\text{post}_{\text{int}}(\mathcal{V}, \ast) &= \{\nu | \exists \nu \in \mathcal{V} : (\nu, \ast, \nu) \in \delta_0\} \\
\text{post}_{\text{ca}}(\mathcal{V}, \ast) &= \{\nu | \exists \nu \in \mathcal{V} : (\nu, \ast, \nu) \in \delta_{ca}\} \\
\text{post}_{\text{re}}(\mathcal{V}, \ast, \ast) &= \{\nu | \exists \nu \in \mathcal{V} : \nu(x) = \nu_c(x_p), \forall \nu_c, \nu \in \mathcal{V}_c : \nu(x) = \nu_c(x_p), \forall \nu_c, \nu, \ast \in \delta_{re}\}
\end{align*}
\]

**Abstract Nested Post Operators** \( \langle \text{post}_{\text{int}}^a, \text{post}_{\text{ca}}^a, \text{post}_{\text{re}}^a \rangle \). The translation of the tuple \( \langle \text{post}_{\text{int}}, \text{post}_{\text{ca}}, \text{post}_{\text{re}} \rangle \) of nested post operators into the triple \( \langle \text{post}_{\text{int}}^a, \text{post}_{\text{ca}}^a, \text{post}_{\text{re}}^a \rangle \) of abstract nested post operators (over bitvectors) is as one expects; see Figure 5.

**Definition 3** (Predicate Automaton \( \mathcal{A}_{\text{Pred}} \)). Given the set of predicates \( \text{Pred} \), the predicate automaton \( \mathcal{A}_{\text{Pred}} \) is the finite nested word automaton where

- the set of states \( Q \) consists of all bitvectors,
- every state is initial, i.e., \( Q_{\text{init}} = Q \),
- every state is final, i.e., \( Q_{\text{fin}} = Q \),
- the triple of transition relations \( \langle \delta_0, \delta_{ca}, \delta_{re} \rangle \) corresponds to the triple \( \langle \text{post}_{\text{int}}^a, \text{post}_{\text{ca}}^a, \text{post}_{\text{re}}^a \rangle \) of abstract nested post operators; i.e.,
  - if \( \ast \) is an assignment or assume statement, then \( (b, \ast, b') \in \delta_0 \) if \( b' \in \text{post}_{\text{int}}^a(b, \ast) \),
  - if \( \ast \) is a call statement, then \( (b, \ast, b') \in \delta_{ca} \) if \( b' \in \text{post}_{\text{ca}}^a(b, \ast) \),
  - if \( \ast \) is a return statement, then \( (b, \ast, b', \ast) \in \delta_{re} \) if \( b' \in \text{post}_{\text{re}}^a(b, \ast, \ast) \).

**Theorem 3.** The predicate automaton \( \mathcal{A}_{\text{Pred}} \) recognizes a superset of the set of feasible nested traces, i.e., \( L(\mathcal{A}_{\text{Pred}}) \supseteq L(\mathcal{A}_S) \).

**Proof.** An accepting run \( \nu_0, \ldots, \nu_n \) of the data automaton \( \mathcal{A}_S \) on a nested trace \( \pi \) gives rise to an accepting run \( b_0, \ldots, b_n \) of the predicate automaton \( \mathcal{A}_{\text{Pred}} \) on \( \pi \) where the \( i \)-th bitvector contains the \( i \)-th valuation, formally \( \nu_i \in [b_i] \) for \( i = 0, \ldots, n \).
\[
\begin{align*}
\text{post}_{\text{in}}(V, \emptyset) &= \{ \nu \cup \{ y \mapsto \nu(t) \} \mid \nu \in V \} \\
\text{post}_{\text{in}}(V, \emptyset) &= \emptyset \\
\text{post}_{\text{ex}}(V, \text{call } p) &= \{ \nu' \mid \{x \mapsto \nu(x)\} \in \nu', \nu \in V \} \\
\text{post}_{\text{en}}(V, \text{call } p) &= \emptyset \\
\text{post}_{\text{ex}}(b, \emptyset) &= \{ b' \mid \text{post}_{\text{ex}}(b, \emptyset) \cap \{ b \} \neq \emptyset \} \\
\text{post}_{\text{in}}(b, \emptyset) &= \{ b' \mid \text{post}_{\text{in}}(b, \emptyset) \cap \{ b \} \neq \emptyset \} \\
\text{post}_{\text{en}}(b, \text{call } p) &= \{ b' \mid \text{post}_{\text{en}}(b, \text{call } p) \cap \{ b \} \neq \emptyset \} \\
\text{post}_{\text{in}}(b, b, \text{call } p) &= \{ b' \mid \text{post}_{\text{in}}(b, \{ b, \text{call } p \}) \cap \{ b \} \neq \emptyset \}
\end{align*}
\]

\textbf{Figure 5.} The triple \( \langle \text{post}_{\text{in}}, \text{post}_{\text{ex}}, \text{post}_{\text{en}} \rangle \) of nested post operators (over sets of valuations \( V \)) is in correspondence with the triple of transition relations of the data automaton \( A_S \). The triple \( \langle \text{post}_{\text{in}}^n, \text{post}_{\text{ex}}^n, \text{post}_{\text{en}}^n \rangle \) of abstract nested post operators (over bitvectors) is in correspondence with the triple of transition relations of the predicate automaton \( A_{\text{Pred}} \).

\section{Nested Interpolant Automata}

We now define a class of finite nested word automata which recognize subsets of the set of infeasible traces. That is, we will use their complement for the nested word automaton \( A \) that serves as proof argument in the proof rule (1) of the previous section.

In many of the settings that we consider, we have a sequence of predicates \( I_0, I_1, \ldots, I_n \) (we refer to the predicates as interpolants for reasons that will become apparent later); this sequence is related to a nested error trace \( \pi \); it may be generated, for example, by the proof of the infeasibility of \( \pi \).

The general notion of an interpolant automaton that we introduce below, however, does not refer to an error trace. It refers to an arbitrary sequence of predicates \( I_0, I_1, \ldots, I_n \). Given such a sequence, we will associate each predicate \( I_i \) with an automaton state \( q_i \). The automaton states are not necessarily pairwise distinct; i.e., we may associate two different predicates \( I_i \) and \( I_j \) with the same automaton state, and we may associate the same predicate with two different states (i.e., we may have \( I_i \neq I_j, q_i = q_j \) and we may have \( I_i = I_j, q_i \neq q_j \)). The non-constructive definition below accommodates a wide range of possible constructions. The definition of a canonical interpolant automaton in Section 5 is constructive.

\textbf{Definition 4 (Nested Interpolant Automaton \( A_{\pi} \)).} Given a sequence of predicates ("interpolants") \( \pi = I_0, I_1, \ldots, I_n \), the nested interpolant automaton

\[ A_{\pi} = (Q_{\pi}, \delta_{\pi}, \delta_{\text{ca}}, \delta_{\text{en}}, Q_{\text{init}}^{\pi}, Q_{\text{fin}}^{\pi}) \]

is a finite nested word automaton if we can index its set of states \( Q_{\pi} \) with the set of indices of the sequence \( 0, \ldots, n \), i.e.,

\[ Q_{\pi} = \{ q_0, \ldots, q_n \} \]

and thus associate each interpolant \( I_i \) with a state \( q_i \), such that the following three conditions hold.

- The sequence of interpolants is inductive wrt. the triple of transition relations \( \langle \delta_{\text{in}}, \delta_{\text{ca}}, \delta_{\text{en}} \rangle \), i.e.,

\[
(q_i, \text{call } p, q_j) \in \delta_{\text{ca}} \Rightarrow \text{post}_{\text{ex}}(I_i, \text{call } p) \subseteq I_j \\
(q_i, \emptyset, q_j) \in \delta_{\text{in}} \Rightarrow \text{post}_{\text{in}}(I_i, \emptyset) \subseteq I_j \\
(q_i, \text{return } p, q_j) \in \delta_{\text{en}} \Rightarrow \text{post}_{\text{en}}(I_i, \text{return } p) \subseteq I_j
\]

- Each interpolant associated with an initial state is the true predicate.

\[ q_i \in Q_{\text{init}}^{\pi} \Rightarrow I_i = \top \]

- Each interpolant associated with a final state is the false predicate.

\[ q_i \in Q_{\text{fin}}^{\pi} \Rightarrow I_i = \bot \]
is used to construct a nested interpolant automaton that recognizes traces infeasible for the same reason (Section 5.4). The two steps are embedded in a counterexample-guided abstraction refinement scheme (Section 5.5).

Before we present the generation of the inductive sequence of Craig interpolants, we give the definition of an inductive sequence of nested interpolants (Section 5.1) and we show how the infeasibility of a nested error trace can be characterized as the satisfiability of a formula (Section 5.2).

5.1 Inductive Sequence of Nested Interpolants

We generalize the concept of a sequence of interpolants [13] to account for the nesting structure. The key idea is that on a return statement the predicate describing the calling context can be combined with the predicate before the return statement that captures the post-condition of the procedure. This is a necessary prerequisite to obtain predicates that only restrict the variables in the current calling context.

**Definition 5** (Inductive Sequence of Nested Interpolants). Given a nested trace
\[ \pi = (s_0 \ldots s_{n-1}, \cdots), \]
an inductive sequence of nested interpolants is a sequence of predicates \( I_0, \ldots, I_n \) such that

- \( I_0 = \top \)
- \( I_n = \bot \)
- For \( i = 0, \ldots, n-1 \):
  - \( \text{post}(I_i, s_i) \subseteq I_{i+1} \) if \( i \) is an internal position,
  - \( \text{post}(I_i, s_i) \subseteq I_{i+1} \) if \( i \) is a call position,
  - \( \text{post}(I_i, s_i) \subseteq I_{i+1} \) if \( i \) is a return position and \( k \rightarrow i \).

Remark. If a nested trace \( \pi \) has an inductive sequence of nested interpolants, then \( \pi \) must be infeasible. In our setting we are only interested in the case were \( \pi \) is an infeasible nested error trace. In Section 5.3 we explain how an inductive sequence of nested interpolants can be obtained from Craig interpolants in that case.

5.2 Infeasibility Proof for Nested Traces

From now on, we assume that the terms in assignment statements and the formulas in assume statements are given in some (first- or higher-order) theory with equality.

For a well-nested trace \( (\pi, \cdots) \) we construct a formula in this theory, called the single static assignment (SSA) for the trace, which is unsatisfiable if and only if the nested trace is infeasible. If the theory is decidable this allows us to obtain a proof of infeasibility for \( (\pi, \cdots) \) automatically, by deciding satisfiability of the SSA.

The single static assignment contains a fresh logical variable for each assignment statement in the trace. This allows us to express these statements by equalities. For recursive programs the variables of different calling contexts have to be kept separate. We use the nesting structure to replace each program variable by the last assigned logical variable that belongs to the same procedure.

**Definition 6** (SSA). The single static assignment (SSA) for a well-nested trace \( \pi = (s_0, \ldots, s_{n-1}, \cdots) \) is the sequence of formulas \( \phi_0, \ldots, \phi_{n-1} \) where \( \phi_i \) is defined as

- \( y^i = \text{rename}_i(t) \) if \( s_i = y^i = t \),
- \( \text{rename}_i(\phi) \) if \( s_i = \phi \),
- \( x^i = \text{rename}_i(x_p) \) if \( s_i = \text{call } p \),
- \( \text{res}_p = \text{rename}_i(\text{res}) \) if \( s_i = \text{return } p \).

The function \( \text{rename}_i \) replaces every variable \( v \) by \( v^i \) where \( j = \text{index}(v, i) \) is the largest index before \( i \) in the same calling context

where \( x \) is assigned. Formally \( \text{index}(v, 0) = -1 \) and

\[
\text{index}(v, i + 1) = \begin{cases} 
  i & \text{if } s_i = y^i = t \text{ and } v = y, \\
  \text{index}(v, i) & \text{if } s_i = \text{return } p \text{ and } v = \text{res}_p, \\
  \text{index}(v, k) & \text{if } k \rightarrow i \text{ (i is return position),} \\
  \text{index}(v, i) & \text{else (i is internal position).}
\end{cases}
\]

The function index is defined recursively and follows the trace from \( i \) backwards to the statement where \( v \) is assigned to. The assignment can be either explicitly by an assignment or implicitly, e.g., when a variable is used uninitialized in a procedure. If a variable is used without prior initialization it returns the index of the procedures call statement and the index \(-1\) in the case where this procedure is the main procedure of the program. The rules for call, return, and internal positions ensure that the index of the variable \( \text{rename}_i(y) \) is in the same procedure as \( i \). To prove the correctness of our interpolation scheme we need the following properties of the renaming function.

**Lemma 1.** The function index in Definition 6 for a well-nested trace \( \pi = (s_0, \ldots, s_{n-1}, \cdots) \) has the following properties:

1. Every variable is assigned before the current position, i.e., \( \text{index}(v, i) < i \).
2. If \( k \rightarrow j \), and \( k < i < j \), then \( k \leq \text{index}(v, i) \), i.e., the variable is assigned inside the current procedure. This also holds for an unfinished call, i.e., \( k \approx \infty \) and \( k < i \).
3. If \( k \rightarrow j \), and \( k < i < j \), i.e., position \( i \) is after the position a procedure returns, then either \( \text{index}(v, i) \geq j \) or \( \text{index}(v, i) < k \), i.e., the variable was not assigned inside that procedure call.
4. If \( s_i = \text{return } p \) then for all \( j > i \) \( \text{index}(y, j) \neq \text{index}(y, i) \), i.e., later statement will never see the value overwritten by statement \( i \).
5. If \( s_i = \text{return } p \) and \( k \) is the corresponding call position (i.e. \( k \approx i \) then \( \text{index}(\text{res}_p, j) \neq \text{index}(\text{res}_p, k) \) for all \( j > i \), i.e., the previous value of the result variable that was overwritten by the return is not visible after the return.

**Proof.**

1. This is easily shown by induction over \( i \).
2. Follows by induction and the fact that the (a) \( \text{index}(v, k + 1) = k \) and (b) there can be no nesting \( k \approx j \) with \( k < i < j \) due to proper nesting.
3. Follows by induction. For the induction step recall that there is no nesting \( k \approx j \) with \( k \leq k' \leq j < j' \) due to proper nesting.
4. By induction: For \( j = i + 1 \) we have \( \text{index}(y, i + 1) = \text{index}(y, j) \). Now assume the hypothesis for \( j > i \) holds. The cases \( \text{index}(y, j + 1) = \text{index}(y, i) \) and \( \text{index}(y, j + 1) = \text{index}(y, k) \) are obvious. If \( \text{index}(y, j + 1) = \text{index}(y, k) \) with \( k \approx j \) there are two cases. If \( k > i \), then \( \text{index}(y, k) \neq \text{index}(y, i) \) by induction hypothesis. Otherwise, \( k < i < j \), hence \( \text{index}(y, k) < k \leq \text{index}(y, i) \). Otherwise the trace is well-nested, \( i \) must be an internal position, so \( k = i \) is not possible. This shows that \( \text{index}(y, j) \neq \text{index}(y, i) \) for all \( j > i \).
5. This is analogous to 4. \( \square \)

We can check the feasibility of a nested trace by checking the satisfiability of its SSA.

**Theorem 5.** Given the SSA \( \phi_0, \ldots, \phi_{n-1} \) for the well-nested trace \( \pi \). The trace \( \pi \) is feasible if and only if the conjunction \( \phi_0 \land \cdots \land \phi_{n-1} \) is satisfiable.

**Proof.** We obtain a run of the data automaton \( A_{\Sigma} \) on the trace \( \pi \) from a model \( M \) of \( \phi_0 \land \cdots \land \phi_{n-1} \) if we set \( \nu_i(v) \) to
The nested trace is infeasible, since the conjunction of the formulas

\[ \psi_7 : x^7 = x^6 \]

\[ \psi_8 : x^7 > 100 \]

Figure 6. The nested trace \( \pi_4 \) (a nested error trace of \( A_{P>0} \)) and its SSA.

\[ M(\text{rename}(v)) \]. Vice versa, we obtain a model from a run \( v_0, \ldots, v_n \) if we assign to \( v \) the value \( v_{i+1}(v) \).

**Example 4.** Consider the nested trace \( \pi_4 \) in Figure 6 and its SSA. The nested trace is infeasible, since the conjunction of the formulas in the SSA is unsatisfiable. In fact

\[ x^7 = x^6 = \text{res}_5^5 = \text{res}_4^1 = x^2 - 10 = x^1 - 10 = x^1 + 1 \leq 101 \]

together with \( x^7 > 100 \) implies \( x^7 = 101 \). This implies

\[ \text{res}_1^{11} = \text{res}_1^{10} = \text{res}_9^9 = x^7 - 10 = 91, \]

which contradicts \( \psi_{12} \).

### 5.3 Inductive Sequence of Nested Craig Interpolants

In this section we give a method to generate an inductive sequence of nested interpolants. The method is based on Craig interpolants, which can be computed automatically from unsatisfiability proofs.

**Craig Interpolant.** Given two formulas \( \psi^- \) and \( \psi^+ \) whose conjunction is unsatisfiable, a Craig interpolant is a formula \( \psi \) such that

- \( \psi^- \) implies \( \psi \),
- the conjunction \( \psi \land \psi^+ \) is unsatisfiable, and
- \( \psi \) contains only variables that occur in both, \( \psi^- \) and \( \psi^+ \).

In the remaining section we assume the SSA is given in a theory where each pair of formulas \( \psi^- \) and \( \psi^+ \) whose conjunction is unsatisfiable has a Craig interpolant. The concept of Craig interpolation can be lifted to a sequence of formulas whose conjunction is unsatisfiable [18] and applied to the SSA of an infeasible trace. For a flat program execution, Craig interpolants derived this way can be used to generate an inductive sequence of interpolants.

If we apply this method to the SSA of a nested trace, a variable of the parent context that is set before calling a procedure and read after the return of the procedure would be allowed to appear in the Craig interpolants for the called procedure. However, this variable is not in the scope of the called procedure and leads to interpolants which are not suitable for a modular analysis. Therefore, we seek Craig interpolants containing only variables that belong to the current calling context.

By moving the instructions of the parent procedure to the end of the sequence as in [14], it is possible to obtain interpolants that are local to the current calling context. We additionally require that the sequence of interpolants is inductive. Therefore, we follow the basic idea of [14] but compute the interpolants from front to back, using the previously computed interpolant as summary that replaces its preceding statements. Instead of conjuncting all preceding statements of the parent procedure we use the previously computed interpolant directly preceding the call statement. This algorithm gives us an inductive sequence of nested interpolants.

**Definition 7 (Inductive Sequence of Nested Craig Interpolants).** Given an SSA \( \phi_0, \ldots, \phi_{n-1} \) and a nesting relation \( \rightsquigarrow \), an inductive sequence of nested Craig interpolants is a sequence \( \psi_0 \ldots \psi_n \), where

- \( \psi_0 \) is \( T \).
- If \( \phi_0, \ldots, \phi_i \) (\( i \in \{0, \ldots, n-1\} \)) are given, we obtain \( \psi_{i+1} \) as follows. We define

\[ \psi_{i+1}^+ = \bigwedge_{j=m+1}^{n-1} \phi_j \land \bigwedge_{k \ll j, k < \min(i+1, j) \neq \infty} (\psi_k \land \phi_k), \]

which is the conjunction of the statements following position \( i+1 \) and the pre-conditions of the calls which are pending at position \( i+1 \), and

\[ \psi_{i+1}^- = \begin{cases} \psi_i \land \phi_i & \text{if } i \text{ is internal position}, \\ \psi_i \land \phi_i & \text{if } i \text{ is call position, } i \rightsquigarrow \infty, \\ \top & \text{if } i \text{ is call position, } i \neq \infty, \\ \psi_i \land \phi_i \land \psi_k \land \phi_k & \text{if } i \text{ is return position, } k \rightsquigarrow i. \end{cases} \]

Then we obtain \( \psi_{i+1} \) as a Craig interpolant of \( \psi_{i+1}^- \) and \( \psi_{i+1}^+ \).

The following lemma is needed for the correctness of the interpolation scheme.

**Lemma 2.** Given an infeasible well-nested trace

\[ \pi = (\phi_0 \ldots \phi_{n-1}, \rightsquigarrow) \]

and its SSA \( \phi_0, \ldots, \phi_{n-1} \).

1. For \( i = 1 \ldots, n \), the conjunction \( \psi_{i-1}^- \land \psi_{i}^+ \) in Definition 7 is unsatisfiable, and therefore the Craig interpolant exists.
2. For \( i = 0 \ldots, n \), the interpolant \( \psi_i \) contains only the variables \( \text{rename}(v) \) for program variables \( v \).
3. For \( i = 0 \ldots, n-1 \),

\[ \psi_i \land \phi_i \Rightarrow \psi_{i+1} \text{ if } i \text{ is an internal or a call position and } \psi_i \land \phi_i \land \psi_k \land \phi_k \Rightarrow \psi_{i+1} \text{ if } i \text{ is a return positions and } k \rightsquigarrow i. \]
4. \( \psi_n = \bot \).

**Proof.** 1. We show this by induction over \( i \). For \( i = 1 \) the conjunction \( \psi_i^- \land \psi_i^+ \) is equal to \( \phi_0 \land \cdots \land \phi_i^- \), which is unsatisfiable by Theorem 5. For the inductive step note that the conjunction \( \psi_{i-1}^- \land \psi_{i}^+ \) has exactly the same conjuncts as \( \psi_i \land \phi_i \). Since \( \psi_i \) was obtained as the Craig interpolant of \( \psi_i^- \land \psi_i^+ \) (which exists by induction hypothesis), the formula \( \psi_i \land \phi_i \) is unsatisfiable.
2. This is also shown by induction. For \( i = 0 \) this is obvious as \( \psi_0 = T \). Now assume that \( \psi_i \land \phi_i \) is unsatisfiable.
rename_e(v) for k ≤ i. All variables appearing in ψ_{i+1} must also appear in ψ_{i+1} and ψ_{i+1}.

If s_i = γ^i, then ψ_{i+1} contains only rename_e(v) and y'. By definition rename_e(v) = rename_e(v) except for v ≡ y. It remains to show that the variable rename_e(y) does not appear in ψ_{i+1}. The formula \( \bigwedge_{j=i+1}^{n} \varphi_j \) contains only rename_e(y) for j > i. Lemma 1.4 ensures rename_e(y) ≠ rename_e(y). The second conjunction of ψ_{i+1} contains the variables rename_e(v), and x_k for procedure calls k → j with k < i + 1 < j < ∞. Lemma 1.2 ensures index(v, k) < k ≤ index(y, i), hence rename_e(v) ≠ rename_e(y). Since the input variable x is never written, x ≡ y.

If s_i = \( \{ γ^i \} \), then by the induction hypothesis and the definition of \( \varphi_i, \psi_{i+1} = \varphi_i \land \varphi_i \) contains only the variables rename_e(v). By definition, rename_e(v) = rename_e(v). Thus, ψ_{i+1} contains only the allowed variables.

If s_i = \{ call p \} is an unfinished call statement, i.e., i →∞, then ψ_{i+1} = \varphi_i \land \varphi_i contains the variables rename_e(v) and x'. The formula ψ_{i+1} contains only variables rename_e(v) with j > i (due to proper nesting there is no call k → j with k ≤ i < j < ∞). By Lemma 1.2 index(v, i) ≤ index(v, j), hence rename_e(v) ≠ rename_e(y). The variable x' = rename_e(x) is allowed to appear in ψ_{i+1}.

If i is a call position with j →∞, then ψ_{i+1} = ⊤ hence ψ_{i+1} does not contain any variable. If s_i = \{ return p \} and k → i is the corresponding call position, then ψ_{i+1} = \varphi_i \land \varphi_i \land \varphi_i \land \varphi_k contains the variables rename_e(v), rename_e(v), x_k, and resp. It is rename_e(v) = rename_e(v) except for v ≡ resp and rename_e(resp) = resp. It remains to show that rename_e(v), rename_e(resp), and x_k do not appear in ψ_{i+1}. The formula ψ_{i+1} contains rename_e(v) for j > i, rename_e(v), and x_k for k' ≥ j with k' < j'. By proper nesting k' < k holds. For j > i we have index(resp, j) ≥ index(resp, k) by Lemma 1.5 and for k' < k < j' Lemma 1.2 gives index(resp, k) ≥ index(resp, k'). Hence rename_e(resp) does not appear in ψ_{i+1}.

The indices of the variables x_k and rename_e(v) are all between k inclusive and i exclusive. For j > i Lemma 1.3 gives index(v, j) ≥ i or index(v, j) < k and for k' < k the index of rename_e(v) and x_k is smaller than k. Therefore, the variables x_k and rename_e(v) do not appear in ψ_{i+1}.

3. These implications follow directly from the property ψ_{i+1} ⇒ ψ_{i+1} of Craig interpolants.

4. Finally, ψ_n = ⊥ follows from ψ_{n+1} = ⊤, because ψ_n \land ψ_{n+1} is unsatisfiable. □

We extend the renaming function to valuations

\[ \text{rename}_e(v) = \{ \text{rename}_e(v) \Rightarrow v \, | \, v \text{ is variable} \} \]

Since the Craig interpolant \( \varphi_i \) contains only rename_e(v) as variables, we can associate it with the set of valuations \( \nu \) such that rename_e(v) = \( \varphi_i \).

**Theorem 6.** Given a well-nested trace \( \pi \), its SSA \( \varphi_0, \ldots, \varphi_{n-1} \), and its inductive sequence of nested Craig interpolants \( \varphi_0, \ldots, \varphi_n \), the sequence of predicates \( I_0, \ldots, I_n \), with

\[ I_i = \{ v \mid \text{rename}_e(v) = \varphi_i \} \text{ for } i = 0, \ldots, n \]

is an inductive sequence of nested interpolants of \( \pi \).

**Proof.** For brevity, we write \( \nu_i \models \varphi_i \) for rename_e(\( \nu_i \)) = \( \varphi_i \), which means \( \nu_i \in I_i \). Similarly, for \( \nu_i, \nu_j \) we write \( \nu_i, \nu_j \models \varphi \) as short-
A
res:=x-10

\[ \text{infeasible error trace} \]

\[ \text{Inductive sequence of nested interpolants} \]

\[ \text{for the nested trace} \]

\[ \text{P} \]

\[ \text{of an inductive sequence of nested interpolants depicted in Figure 7. For better legibility we omitted the edges} \]

\[ (q_0, \text{res:=x-10}, q_4), (q_1, \text{res:=x-10}, q_5), (q_{10}, q_2, \text{return}, q_0), \]

\[ \text{which are not needed to prove correctness} \]

\[ \text{of} \]

\[ P^{91} \]

\[ \text{Example 5. Figure 7 shows the sequence of nested interpolants for the nested trace} \]

\[ \text{π} \]

\[ \text{π} \]

\[ \text{for} \]

\[ \text{the} \]

\[ \text{infeasibility we omitted the edges} \]

\[ (q_0, \text{res:=x-10}, q_4), (q_1, \text{res:=x-10}, q_5), (q_{10}, q_2, \text{return}, q_0), \]

\[ \text{which are not needed to prove correctness} \]

\[ \text{of} \]

\[ P^{91} \]

\[ \text{Example 6. Figure 8 shows the canonical interpolant automaton} \]

\[ A^\pi_2 \]

\[ \text{for the inductive sequence of nested interpolants} \]

\[ \text{depicted in Figure 6. The interpolants were computed as Craig interpolants.} \]

\[ \text{5.4 Canonical Interpolant Automaton} \]

\[ \text{We use an inductive sequence of nested interpolants for a nested} \]

\[ \text{error trace} \]

\[ \text{π} \]

\[ \text{for} \]

\[ \text{a nested interpolant automaton.} \]

\[ \text{The following construction accepts} \]

\[ \text{π} \]

\[ \text{and other traces that are infeasible} \]

\[ \text{for the same reason as} \]

\[ \text{π} \]

\[ \text{Definition 8 (Canonical Interpolant Automaton} A^\pi_2). \]

\[ \text{Given an inductive sequence of nested interpolants} \]

\[ \text{I} = I_0, I_1, \ldots, I_n \]

\[ \text{of an infeasible error trace} \]

\[ \text{π} = (s_0, s_1, \ldots, s_{n-1}, \text{∞}) \]

\[ \text{along the sequence of locations} \]

\[ \ell_0, \ldots, \ell_n \]

\[ \text{(the run of the program automaton), we introduce pairwise different states} \]

\[ q_0, \ldots, q_n \]

\[ \text{and define the canonical} \]

\[ \text{interpolant automaton} A^\pi_2 \]

\[ \text{for} \]

\[ \text{π} \]

\[ \text{as follows.} \]

\[ A^\pi_2 = \langle Q, (\delta_n, \delta_{ca}, \delta_e), Q^{\min}, Q^{\max} \rangle \]

\[ \bullet Q^r = \{q_0, \ldots, q_n\} \]

\[ \delta_n = \{(q_j, s_j, s_{j+1}) | j \text{ is an internal position}, i, j = 0, \ldots, n-1, j \leq i, \ell_i = \ell_j, \text{post}_{ca}(I_i, s_j) \subseteq I_{j+1}\} \]

\[ \delta_{ca} = \{(q_j, s_j, s_{j+1}) | j \text{ is a call position}, i, j = 0, \ldots, n-1, j \leq i, \ell_i = \ell_j, \text{post}_{ca}(I_i, s_j) \subseteq I_{j+1}\} \]

\[ \delta_e = \{(q_{i+1}, q_i, s_j, s_{j+1}) | j \text{ is a return position, } k \sim j \}

\[ \bullet Q^{\min}_e = \{q_0\} \]

\[ \bullet Q^{\max}_e = \{q_n\} \]

\[ \text{Figure 8. Canonical interpolant automaton} A^\pi_2 \]

\[ \text{for the inductive} \]

\[ \text{sequence of nested interpolants} \]

\[ \text{depicted in Figure 7. For better legibility we omitted the edges} \]

\[ (q_0, \text{res:=x-10}, q_4), (q_1, \text{res:=x-10}, q_5), (q_{10}, q_2, \text{return}, q_0), \]

\[ \text{which are not needed to prove correctness} \]

\[ \text{of} \]

\[ P^{91} \]

\[ \text{Example 6. Figure 8 shows the canonical interpolant automaton} \]

\[ A^\pi_2 \]

\[ \text{for the} \]

\[ \text{π} \]

\[ \text{and its inductive sequence of nested interpolants in Figure 7.} \]

\[ \text{The interpolant automaton is not sufficient to prove the correctness of the program using proof rule (1). The automaton} A^\pi_2 \]

\[ \text{does not accept the nested error trace} \]

\[ \text{π} \]

\[ \text{from Figure 4. Hence, for the complement automaton} A^\pi_2 \]

\[ \text{the precondition of proof rule (1)} \]

\[ \text{does not hold.} \]

\[ \text{Figure 9 shows a second interpolant automaton} A^\pi_3 \]

\[ \text{that accepts} \]

\[ \text{π} \]

\[ \text{and infeasible nested traces similar to} \]

\[ \text{π} \]

\[ \text{The product automaton of the complemented interpolant automata} A^\pi_2 \]

\[ \text{still recognizes a superset of feasible traces} \]

\[ \text{A} \]

\[ \text{It can be mechanically checked that} \]

\[ \text{which proves the correctness of} \]

\[ P^{91} \]

\[ \text{with proof rule (1).} \]

\[ 5.5 \text{Counterexample-Guided Abstraction Refinement} \]

\[ \text{The example of the last section motivates the iterated refinement scheme depicted in Figure 10, similar to the classical check-}

\[ \text{analyze-refine loop [5, 10].} \]

\[ \text{We start with a coarse abstraction} \ A \]

\[ \text{of the data automaton} A_{\Sigma}, \]

\[ \text{e.g., the automaton that accepts any nested trace. If the abstraction is not a proof for correctness, it accepts a nested error trace, say} \]

\[ \text{π} \]

\[ \text{We check whether} \]

\[ \text{π} \]

\[ \text{is infeasible. If this is the case, we obtain a} \]

\[ \text{nested interpolant automaton} A_2 \]

\[ \text{that accepts} \]

\[ \text{π} \]

\[ \text{This automaton can be the canonical interpolant automaton} A^\pi_2. \]

\[ \text{We refine our} \]
abstraction by

\[ A := A \cap \overline{A} \]

and repeat the loop with the new abstraction.

If we construct a deterministic interpolant automaton \( A', \) complementing the automaton boils down to inverting the set of final states. The intersection of the program automaton and the interpolant automaton can be computed by unfolding the program automaton, annotating the nodes with the interpolants of the interpolant automaton and merging nodes with the location and same interpolant. This is essentially an extension of the algorithm from [19] to recursive programs.

![Diagram](image-url)

**Figure 10.** Counterexample-guided abstraction refinement scheme that exploits proof rule (1). If \( \mathcal{L}(A) \supseteq \mathcal{L}(A_S) \) and \( \mathcal{L}(A) \cap \mathcal{L}(A_P) = \emptyset \) then program \( P \) is correct.

### 6. Conclusion

In this paper, we have explored the potential of the theory of nested words as a foundation for correctness proofs for the general class of recursive procedures. Our conceptual contribution is a simple framework that allows us to shine a new light on classical concepts such as Floyd/Hoare proofs and predicate abstraction for recursive programs. Our technical contribution is to give, to our knowledge for the first time, a principled method that constructs an abstract proof for recursive programs from interpolants (avoiding the construction of the abstract transformer). We have used nested words to formalize the concept of inductive sequences of interpolants for traces of recursive programs. The construction of nested word automata from inductive sequences interpolants is interesting because it avoids the construction of the abstract transformer. As pointed out in [19], the interpolation-based proof method can be made to scale if one exploits the modular structure of procedural programs (i.e., using procedure summaries). Our work provides the foundation to explore different realizations of this approach.

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### References


