Geometric Series as Nontermination Arguments for Linear Lasso Programs

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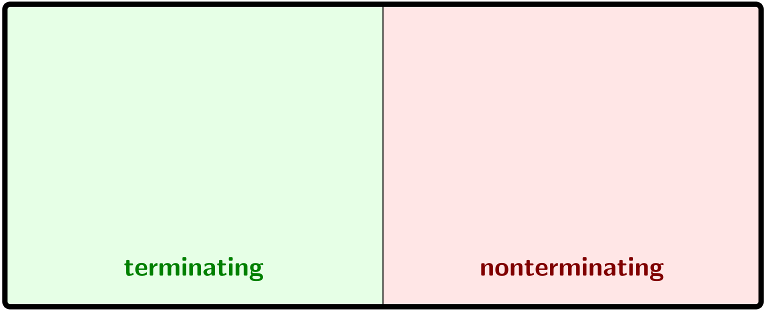
Nontermination Analysis

nonterminating == nonterminating for some input
== at least one infinite execution



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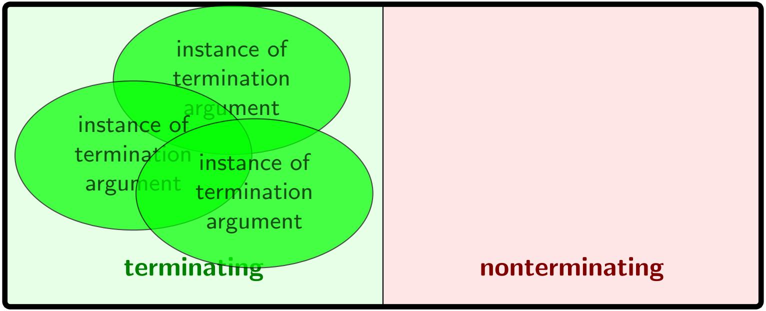

Kinds of Termination Arguments

- ranking function
- transition invariant
- size-change graphs
- dependency pair
- ...

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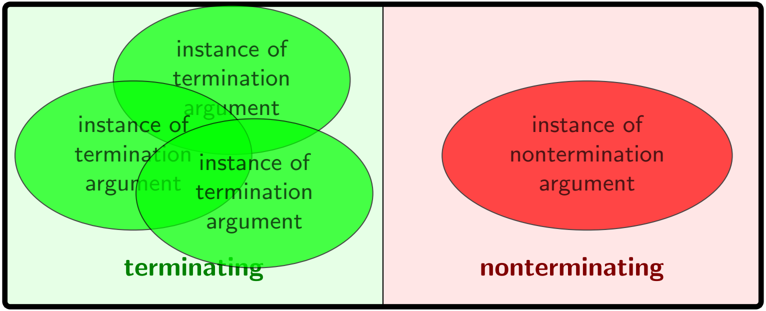

Kinds of Termination Arguments

- ranking function
- transition invariant
- size-change graphs
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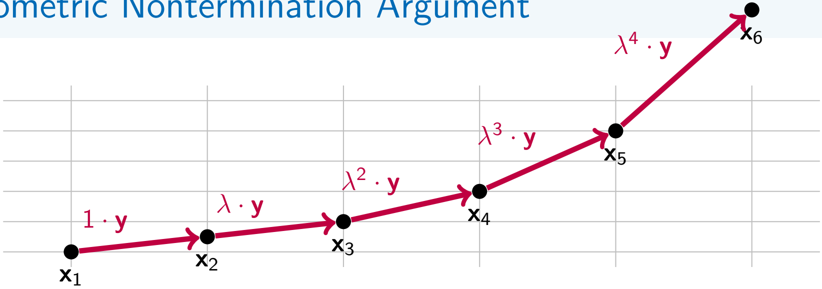
- ...

Kinds of Nontermination Arguments

- recurrence set
- underaproximation which is nonterminating for each input
- ...
- geometric nontermination argument

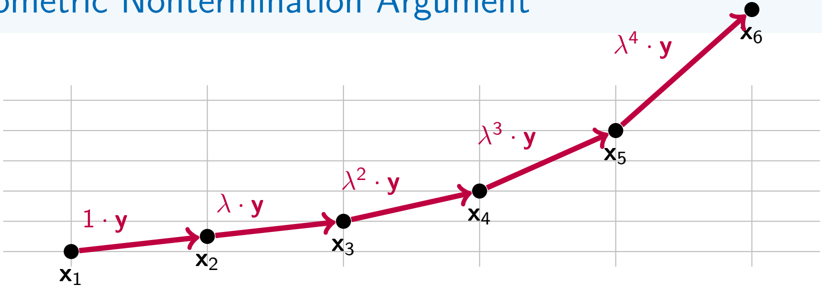
Geometric Nontermination Argument



witness for existence of infinite execution (of the following form)

$$\mathbf{x}_0, \qquad \mathbf{x}_1, \qquad \mathbf{x}_1 + \mathbf{y}, \qquad \mathbf{x}_1 + (1+\lambda)\cdot\mathbf{y}, \qquad \mathbf{x}_1 + (1+\lambda+\lambda^2)\cdot\mathbf{y}, \qquad \ldots$$

Geometric Nontermination Argument



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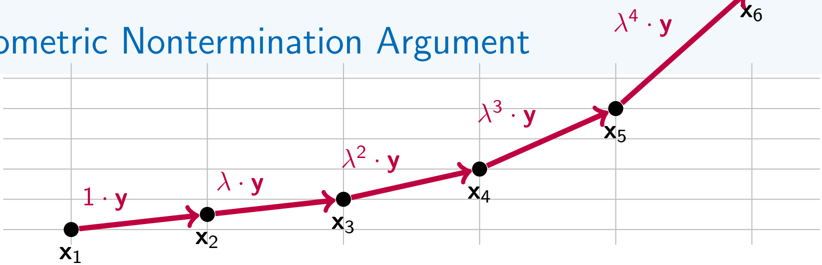

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useful in practice

- Benchmark set from

  Brockschmidt, Cook, Fuhs **Better termination proving through cooperation** (CAV 2013)

  contains 181 programs whose nontermination is known, our tool can prove nontermination for 170 of them
- Benchmarks set from Termination Competition 2014

Lasso Program $P = (\text{STEM}, \text{LOOP})$

A *lasso program* P consists of two binary relations $\text{STEM}(\mathbf{x}, \mathbf{x'})$ and $\text{LOOP}(\mathbf{x}, \mathbf{x'})$ over a set of states.
A sequence of states $\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4 \ldots$ is called an *infinite execution* if

- $(\mathbf{s}_0, \mathbf{s}_1) \in \text{STEM}$, and
- $(\mathbf{s}_t, \mathbf{s}_{t+1}) \in \text{LOOP}$ for all $t \geq 1$.

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Example

```
b := b - 1
while (a ≥ 0) {
    a := a - b
}
```

$\textsc{stem}(\binom{a}{b}, \binom{a'}{b'})$
$b' = b - 1 \land a' = a$

$\textsc{loop}(\binom{a}{b}, \binom{a'}{b'})$
$a \geq 0 \land a' = a - b \land b' = b$

Infinite execution

$\binom{42}{1}, \binom{42}{0}, \binom{42}{0}, \binom{42}{0}, \binom{42}{0}, \ldots$

Preliminary Considerations

a simple case

The lasso program $P = (\textsc{stem}, \textsc{loop})$ has an execution of the form

$$\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_1, \mathbf{s}_1, \mathbf{s}_1 \ldots$$

iff the following formula is satisfiable.

$$\textsc{stem}(\mathbf{s}_0, \mathbf{s}_1) \quad \wedge \quad \textsc{loop}(\mathbf{s}_1, \mathbf{s}_1)$$

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$$a \geq 0 \wedge a' = a - b \wedge b' = b$$

$$\begin{matrix} a_0 \mapsto 42 & a_1 \mapsto 42 \\ b_0 \mapsto 1 & b_1 \mapsto 0 \end{matrix}$$ is satisfying assignment

A “difficult” program

```
while (a ≥ 2) {
     a := 2*a + 1
}
```

$$a_0 = 2,\ a_1 = 2,\ a_2 = 5,\ a_3 = 11,\ a_4 = 23,\ a_5 = 47,\ a_6 = 95,\ a_7 = 191,\ \ldots$$

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Consider only lasso programs whose relations STEM and LOOP are given by a conjunction of linear inequalities over the reals.

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relation $\textsc{loop}(a, a')$

$$\begin{pmatrix} -1 & 0 \\ -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ a' \end{pmatrix} \leq \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

$$a_0 = 2,\ a_1 = 2,\ a_2 = 5,\ a_3 = 11,\ a_4 = 23,\ a_5 = 47,\ a_6 = 95,\ a_7 = 191,\ \ldots$$

Consider only lasso programs whose relations STEM and LOOP are given by a conjunction of linear inequalities over the reals.
We use vectors and matrices to denote conjunctsions of linear inequalities. $A \left(\begin{smallmatrix} \mathbf{x} \\ \mathbf{x}' \end{smallmatrix} \right) \leq \mathbf{b}$

Geometric Nontermination Argument

Let $P = (\text{STEM}, \text{LOOP})$ be a linear lasso program such that LOOP is defined by the formula $A\binom{\mathbf{x}}{\mathbf{x}'} \leq \mathbf{b}$. The tuple $N = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{y}, \lambda)$ is called a *geometric nontermination argument* for P iff the following properties hold.

(domain) $\mathbf{x}_0, \mathbf{x}_1, \mathbf{y} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $\lambda > 0$.

(init) $(\mathbf{x}_0, \mathbf{x}_1) \in \text{STEM}$

(point) $A\binom{\mathbf{x}_1}{\mathbf{x}_1 + \mathbf{y}} \leq \mathbf{b}$

(ray) $A\binom{\mathbf{y}}{\lambda\mathbf{y}} \leq \mathbf{0}$

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&\text{(point)} && A \binom{\mathbf{x}_1}{\mathbf{x}_1 + \mathbf{y}} \leq \mathbf{b} \\
&\text{(ray)} && A \binom{\mathbf{y}}{\lambda \mathbf{y}} \leq \mathbf{0}
\end{aligned}
$$

Theorem (Soundness)

If the conjunctive linear lasso program $P = (\textsc{stem}, \textsc{loop})$ has a geometric nontermination argument $N = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{y}, \lambda)$ then P has the following infinite execution.

$$\mathbf{x}_0, \quad \mathbf{x}_1, \quad \mathbf{x}_1 + \mathbf{y}, \quad \mathbf{x}_1 + (1+\lambda) \cdot \mathbf{y}, \quad \mathbf{x}_1 + (1+\lambda+\lambda^2) \cdot \mathbf{y}, \quad \ldots$$

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We obtain $N = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{y}, \lambda)$ via constraint solving

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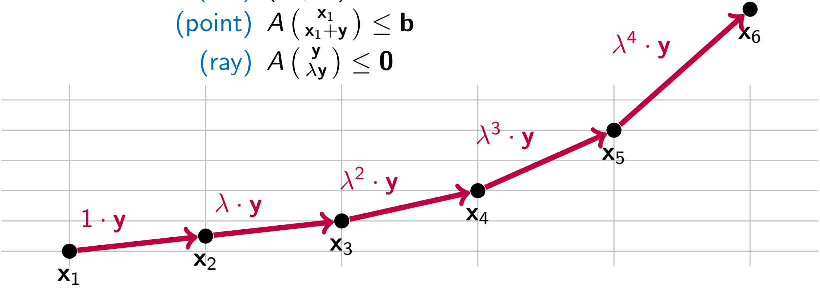

$$\mathbf{x}_0, \quad \mathbf{x}_1, \quad \mathbf{x}_1 + \mathbf{y}, \quad \mathbf{x}_1 + (1+\lambda)\cdot\mathbf{y}, \quad \mathbf{x}_1 + (1+\lambda+\lambda^2)\cdot\mathbf{y}, \quad \ldots$$

```
while (a ≥ 2) {
    a := 2*a + 1
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```

relation $\textsc{loop}(a, a')$

$$\begin{pmatrix} -1 & 0 \\ -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ a' \end{pmatrix} \leq \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

Constraints for Geometric Nontermination Argument

(domain) $\mathbf{x}_0, \mathbf{x}_1, \mathbf{y} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $\lambda > 0$.

(init) $(\mathbf{x}_0, \mathbf{x}_1) \in \textsc{stem}$

(point) $A \left(\begin{smallmatrix} \mathbf{x}_1 \\ \mathbf{x}_1 + \mathbf{y} \end{smallmatrix} \right) \leq \mathbf{b}$

(ray) $A \left(\begin{smallmatrix} \mathbf{y} \\ \lambda \cdot \mathbf{y} \end{smallmatrix} \right) \leq \mathbf{0}$

For $a_0 = 2$, $a_1 = 2$, $y = 3$ and $\lambda = 2$, the tuple $N = (a_0, a_1, y, \lambda)$ is a geometric nontermination argument and the following sequence of states is an infinite execution of P.

$a_0 = 2,\ a_1 = 2,\ a_2 = 5,\ a_3 = 11,\ a_4 = 23,\ a_5 = 47,\ a_6 = 95,\ a_7 = 191,\ \ldots$

Theorem (Soundness)

If the conjunctive linear lasso program $P = (\textsc{stem}, \textsc{loop})$ *has a geometric nontermination argument* $N = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{y}, \lambda)$ *then* P *has the following infinite execution.*

$$\mathbf{x}_0,\ \mathbf{x}_1,\ \mathbf{x}_1 + \mathbf{y},\ \mathbf{x}_1 + (1+\lambda)\mathbf{y},\ \mathbf{x}_1 + (1+\lambda+\lambda^2)\mathbf{y},\ \ldots$$

Proof.

Define $\mathbf{z}_0 := \mathbf{x}_0$ and $\mathbf{z}_t := \mathbf{x}_1 + \sum_{i=0}^{t} \lambda^i \mathbf{y}$. Then $(\mathbf{z}_t)_{t \geq 0}$ is an infinite execution of P: by (init), $(\mathbf{z}_0, \mathbf{z}_1) = (\mathbf{x}_0, \mathbf{x}_1) \in \textsc{stem}$ and

$$A \begin{pmatrix} \mathbf{z}_t \\ \mathbf{z}_{t+1} \end{pmatrix} = A \begin{pmatrix} \mathbf{x}_1 + \sum_{i=0}^{t} \lambda^i \mathbf{y} \\ \mathbf{x}_1 + \sum_{i=0}^{t+1} \lambda^i \mathbf{y} \end{pmatrix} = A \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_1 + \mathbf{y} \end{pmatrix} + \sum_{i=0}^{t} \lambda^i A \begin{pmatrix} \mathbf{y} \\ \lambda \mathbf{y} \end{pmatrix} \leq \mathbf{b} + \sum_{i=0}^{t} \lambda^i \mathbf{0} = \mathbf{b},$$

by (point) and (ray). □

infinite execution

$$\mathbf{x}_0, \qquad \mathbf{x}_1, \qquad \mathbf{x}_1+\mathbf{y}, \qquad \mathbf{x}_1+(1+\lambda)\cdot\mathbf{y}, \qquad \mathbf{x}_1+(\underbrace{1+\lambda+\lambda^2}_{\text{geometric series}})\cdot\mathbf{y}, \qquad \ldots$$

closed formula

for $i \geq 2$ $\quad \mathbf{x}_i = \mathbf{x}_1 + \dfrac{\lambda^{i+1}-1}{\lambda-1}\cdot\mathbf{y}$

Example

The following linear lasso program has an infinite execution, e.g. $\begin{pmatrix} 2^i \\ 3^i \end{pmatrix}_{i \geq 0}$, but it does not have a geometric nontermination argument.

```
while (a ≥ 1 && b ≥ 1 ) {
     a := 2*a
     b := 3*b
}
```

Let $|\cdot| : \mathbb{R}^n \to \mathbb{R}$ denote some norm. We call an infinite execution $(\mathbf{x}_t)_{t \geq 0}$ *bounded* iff there is a real number $d \in \mathbb{R}$ such that for each state its norm in bounded by d, i.e. $|\mathbf{x}_t| \leq d$ for all t.

Lemma (Fixed Point)

Let $P = (\text{STEM}, \text{LOOP})$ be a linear loop program such that $\text{STEM} = id$. The loop P has a bounded infinite execution if and only if there is a fixed point $\mathbf{x}^* \in \mathbb{R}^n$ such that $(\mathbf{x}^*, \mathbf{x}^*) \in \text{LOOP}$.

Corollary

If the linear loop program $P = (id, \text{LOOP})$ has a bounded infinite execution, then it has a geometric nontermination argument.

Recurrence Set

A recurrence set S is a set of states such that

- at least one state of S is in the range of STEM, i.e.

$$\exists \mathbf{x}, \mathbf{x}'.\ (\mathbf{x}, \mathbf{x}') \in \text{STEM} \land \mathbf{x'} \in S, \text{ and}$$

- for each state in S there is at least one LOOP-successor that is in S, i.e.,

$$\forall \mathbf{x}.\ \mathbf{x} \in S \rightarrow \exists \mathbf{x}'.\ (\mathbf{x}, \mathbf{x}') \in \text{LOOP} \land \mathbf{x'} \in S.$$

If we restrict the form of S to a convex polyhedron, (i.e. $S = \bigwedge_i \mathbf{a}_i \cdot \mathbf{x} \geq d_i$)
we can encode its existence using algebraic constraints.

Gupta, Henzinger, Majumdar, Rybalchenko, Xu **Proving non-termination** (POPL 2008)

Rybalchenko **Constraint solving for program verification theory and practice by example** (CAV 2010)

Recurrence Set

Lemma

Let $P = (\text{STEM}, \text{LOOP})$ *be a linear lasso program and* $N = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{y}, \lambda)$ *be a geometric nontermination argument for* P*. The following set* S *is a recurrence set for* P*.*

$$S = \left\{ \mathbf{x}_1 + \sum_{i=0}^{t} \lambda^i \mathbf{y} \mid t \in \mathbb{N} \right\}$$

Integers vs. Reals

Terminating over the Reals $\Rightarrow$ Terminating over the Integers

Constraints for Geometric Nontermination Argument

$$\begin{aligned}
\text{(domain)} &\quad \mathbf{x}_0, \mathbf{x}_1, \mathbf{y} \in \mathbb{R}^n,\ \lambda \in \mathbb{R} \text{ and } \lambda > 0. \\
\text{(init)} &\quad (\mathbf{x}_0, \mathbf{x}_1) \in \textsc{stem} \\
\text{(point)} &\quad A \binom{\mathbf{x}_1}{\mathbf{x}_1 + \mathbf{y}} \leq \mathbf{b} \\
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\end{aligned}$$

Future Work

- If LOOP is linear update and STEM is identity then termination is decideable.

Ashish Tiwari	**Termination of linear programs**	(CAV 2004)

Mark Braverman	**Termination of integer linear programs**	(CAV 2006)

  Approach: analyze eigenvalues

- Our approach: relations LOOP and STEM given by linear constraints

Can we combine both approaches?

Our tool: LassoRanker

http://ultimate.informatik.uni-freiburg.de/LassoRanker/