In the lecture on Wednesday, we will introduce a programming language and define its semantics formally. For defining the semantics we will use the reflexive transitive closure of a relation. This exercise sheet should make you familiar with that term.

Given a sets $X$ a binary relation over $X$ is a subset of the Cartesian product $X \times X$. We call a binary relation $R$ reflexive if for all $x \in X$ the pair $(x, x)$ is an element of $R$ and we call a binary relation $R$ transitive if for all $x, y, z \in X$ the following property holds.

$$
\text{If } (x, y) \in R \text{ and } (y, z) \in R \text{ then } (x, z) \in R.
$$

**Exercise 1: Reflexivity and Transitivity**

State for each of the following relations if the relation is reflexive and if the transition is transitive.

(a) The “strictly smaller” relation over Integers. $R_a = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x < y\}$

(b) The relation $\text{win}_{RPS}$ over the set $\{\text{Rock, Paper, Scissors}\}$.
   $$
   R_b = \{(\text{Rock, Scissors}), (\text{Scissors, Paper}), (\text{Paper, Rock})\}
   $$

(c) $R_c = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (x - y) \mod 42 = 0\}$

(d) $R_d = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (x - y) = 2 \text{ or } (y - x) = 2\}$

Given a binary relation $R$ over the set $X$, the reflexive transitive closure, denoted $R^*$, is the smallest relation such that $R \subseteq R^*$, $R^*$ is reflexive and $R^*$ is transitive. We note that in this context “smallest” means that there is no strict subset that has the same properties and that this minimum is indeed unique since reflexive and transitive relations are closed under intersection.

**Exercise 2: Reflexive Transitive Closure**

Compute for each of the four relations above the reflexive transitive closure.