Dr. Matthias Heizmann
Tanja Schindler
Dominik Klumpp

Hand in until 16:15 on May 15, 2019
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Tutorial for Program Verification
Exercise Sheet 5

In the lecture on Wednesday, we will introduce a programming language and define its semantics formally. For defining the semantics we will use the *reflexive transitive closure* of a relation. This exercise sheet should make you familiar with that term.
Given a sets X a *binary relation* over X is a subset of the Cartesian product $X \times X$.
We call a binary relation R *reflexive* if for all $x \in X$ the pair (x, x) is an element of R and we call a binary relation R *transitive* if for all $x, y, z \in X$ the following property holds.

$$\text{If } (x, y) \in R \text{ and } (y, z) \in R \text{ then } (x, z) \in R.$$

Exercise 1: Reflexivity and Transitivity — 1 Point

State for each of the following relations if the relation is reflexive and if the transition is transitive.

(a) The "strictly smaller" relation over Integers. $R_a = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x < y\}$

(b) The relation $\mathsf{win}_{\mathsf{RPS}}$ over the set $\{\mathsf{Rock}, \mathsf{Paper}, \mathsf{Scissors}\}$.

$R_b = \{(\mathsf{Rock}, \mathsf{Scissors}), (\mathsf{Scissors}, \mathsf{Paper}), (\mathsf{Paper}, \mathsf{Rock})\}$

(c) $R_c = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (x - y) \mod 42 = 0\}$

(d) $R_d = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (x - y) = 2 \text{ or } (y - x) = 2\}$

Given a binary relation R over the set X, the *reflexive transitive closure*, denoted R^*, is the smallest relation such that $R \subseteq R^*$, R^* is reflexive and R^* is transitive.
We note that in this context "smallest" means that there is no strict subset that has the same properties and that this minimum is indeed unique since reflexive and transitive relations are closed under intersection.

Exercise 2: Reflexiv Transitive Closure — 2 Points

Compute for each of the four relations above the reflexiv transitive closure.