Exercise 1: Soundness of the Weakening Postcondition Rule 1 Point
Prove that the weakening postcondition rule of the Hoare proof system displayed below is sound.

\[
(\text{weakpos}) \quad \{\varphi\} st \{\psi\} \quad \text{if} \quad \psi \models \psi'
\]

More precisely, prove the following lemma from the lecture:

If the Hoare triple \(\{\varphi\} st \{\psi\}\) is valid and the side condition \(\psi \models \psi'\) is valid, then the Hoare triple \(\{\varphi\} st \{\psi'\}\) is valid.

Exercise 2: Soundness of the Composition Rule 2 Bonus Points
Prove that the composition rule of the Hoare proof system displayed below is sound.

\[
(\text{compo}) \quad \{\varphi_1\} st_1 \{\varphi_2\} \quad \{\varphi_2\} st_2 \{\varphi_3\} \\
\{\varphi_1\} st_1 st_2 \{\varphi_3\}
\]

More precisely, prove the following lemma from the lecture:

If the Hoare triple \(\{\varphi_1\} st_1 \{\varphi_2\}\) is valid and the Hoare triple \(\{\varphi_2\} st_2 \{\varphi_3\}\) is valid, then the Hoare triple \(\{\varphi_1\} st_1 st_2 \{\varphi_3\}\) is valid.

Exercise 3: Soundness of the Conditional Rule 2 Points
Prove that the conditional rule of the Hoare proof system displayed below is sound.

\[
(\text{condi}) \quad \{\varphi \land expr\} st_1 \{\psi\} \\
\{\varphi \land \neg expr\} st_2 \{\psi\} \\
\{\varphi\} \text{if(expr)} \{st_1\} \text{else}\{st_2\} \{\psi\}
\]

More precisely, prove the following lemma from the lecture:

If the Hoare triple \(\{\varphi \land expr\} st_1 \{\psi\}\) is valid and the Hoare triple \(\{\varphi \land \neg expr\} st_2 \{\psi\}\) is valid, then the Hoare triple \(\{\varphi\} \text{if(expr)} \{st_1\} \text{else}\{st_2\} \{\psi\}\) is valid.
Exercise 4: Hoare Logic Proof

Consider the following Boo program $P = (V, \mu, st_P)$ with $V = \{i, j, x, y\}$, $\mu(i) = \mu(j) = \mu(x) = \mu(y) = \mathbb{Z}$, and $st_P$ (a derivation tree of the Boo grammar for) the program code shown below.

```
x := i;
y := j;
while (x != 0) {
x := x - 1;
y := y - 1
}
```

Give a Hoare logic proof that shows that $\{\text{true}\} \triangleright st_P \{i = j \rightarrow y = 0\}$ is a valid Hoare triple.

Exercise 5: Satisfiability in the Theory of Arrays

Determine which of the following FOL formulas is satisfiable in the theory of arrays. If a formula is satisfiable, give a satisfying assignment. You may assume that the arrays have integer indices and values.

(a) $\text{select}(a, i) = i \land \text{store}(a, i, k) = a \land i \neq k$

(b) $a = \text{store}(b, k, v) \land \text{select}(a, i) \neq \text{select}(b, i) \land \text{select}(a, j) \neq \text{select}(b, j)\$

(c) $b = \text{store}(a, k, v) \land \forall i. i \neq j \rightarrow \text{select}(a, i) = \text{select}(b, i)\$

You may use an SMT solver to solve this task. To declare an array constant $a$, you can use the SMT-LIB command `(declare-fun a () (Array Int Int))`. The function applications for the `select` function and the `store` function are written as usual, e.g. `(select a i)` and `(store a i v)`.

Exercise 6: Theory of Arrays

Formalize the following statements as first order logic formulas.

(a) The array $a$ has the value 0 at every index except at index 5, where the value is 23.

(b) The array $a$ contains no duplicate values between the indices 0 and 10 inclusive.