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Hand in until 16:00 on April 28, 2021
Discussion: April 28, 2021

Tutorial for Program Verification
Exercise Sheet 2

In this exercise sheet we work with syntax and semantics of *First-Order Logic*. Also, the bonus exercise gives you another chance to practice how to prove implications with the $\mathcal{N}_{\mathsf{PL}}$ proof system for propositional logic.

Submit your solution by uploading it as PDF in ILIAS.

Exercise 1: First-Order Logic: Vocabularies 3 Points

(a) State a vocabulary such that the number of *FOL* terms is finite but not zero.

(b) State a vocabulary such that the number of *FOL* terms is infinite.

(c) How many *FOL* formulas do we have if the vocabulary is $\mathcal{V} = (\emptyset, \emptyset, \emptyset, \emptyset)$?

Exercise 2: Models for Quantifier-free Formulas 3 Points

Consider the vocabulary $\mathcal{V} = (\{x, y, z\}, \emptyset, \{f, g\}, \{p\})$ and the following formula.

$$\varphi: \; p(f(x, y), z) \rightarrow p(y, g(z, x))$$

(a) Consider the model $\mathcal{M} = (\mathcal{D}, \mathcal{I})$, where $\mathcal{D} = \mathbb{Z}$ and $\mathcal{I}$ maps f to the addition function ("+"), g to the subtraction function ("-"), and p to the strictly smaller relation ("$<$"). Consider the variable assignment ρ such that $\rho(x) = 13$, $\rho(y) = 42$, and $\rho(z) = 1$.

  (i) What is $[\![\varphi]\!]_{\mathcal{M},\rho}$? Show also intermediate results for at least three different subterms or subformulas.

  (ii) Let us define ρ' as $\rho \lhd \{x \mapsto 55\}$ What is $[\![\varphi]\!]_{\mathcal{M},\rho'}$?

  (iii) State an interpretation function $\mathcal{I}'$ such that for the model $\mathcal{M}' = (\mathcal{D}, \mathcal{I}')$ the truth value $[\![\varphi]\!]_{\mathcal{M}',\rho}$ is different from the truth value $[\![\varphi]\!]_{\mathcal{M},\rho}$.

  (iv) State a formula φ' that also contains the symbols x, y, z, f, g, p such that the truth value $[\![\varphi']\!]_{\mathcal{M},\rho}$ is different from the truth value $[\![\varphi]\!]_{\mathcal{M},\rho}$.

(b) Consider the interpretation domain $\mathcal{D}_{\mathsf{RPS}} = \{\mathsf{Rock}, \mathsf{Paper}, \mathsf{Scissors}\}$, the function $\mathsf{fst} : \mathcal{D}_{\mathsf{RPS}} \times \mathcal{D}_{\mathsf{RPS}} \rightarrow \mathcal{D}_{\mathsf{RPS}}$ that is defined as $\mathsf{fst}(x, y) = x$ for all $x, y \in \mathcal{D}_{\mathsf{RPS}}$ and the binary relation $R_{\mathsf{win}} \subseteq \mathcal{D}_{\mathsf{RPS}} \times \mathcal{D}_{\mathsf{RPS}}$ which is defined as follows.

$$R_{\mathsf{win}} = \{(\mathsf{Rock}, \mathsf{Scissors}), (\mathsf{Scissors}, \mathsf{Paper}), (\mathsf{Paper}, \mathsf{Rock})\}$$

Let $\mathcal{M} = (\mathcal{D}_{\mathsf{RPS}}, \mathcal{I})$ be the model whose interpretation function $\mathcal{I}$ maps f to fst, g to fst, and p to R_{win}. Let ρ be the variable assignment that is defined as follows $\rho(x) = \mathsf{Rock}$, $\rho(y) = \mathsf{Paper}$, $\rho(z) = \mathsf{Scissors}$.

(i) What is $[\![\varphi]\!]_{\mathcal{M},\rho}$? Show also intermediate results for at least three different subterms or subformulas.

(ii) State a formula φ' that also contains the symbols x, y, z, f, g, p such that the truth value $[\![\varphi']\!]_{\mathcal{M},\rho}$ is different from the truth value $[\![\varphi]\!]_{\mathcal{M},\rho}$.

Exercise 3: Propositional Logic: Natural Deduction Proof 2 Bonus Points

Prove the following implication in the Natural Deduction proof system $\mathcal{N}_{\mathsf{PL}}$.

$$\{B \wedge C\} \vDash A \rightarrow B$$