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Hand in until 10:00 on May 03, 2021
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Tutorial for Program Verification
Exercise Sheet 3

In this exercise we work with syntax and semantics of First-Order Logic (FOL), and we practice conducting proofs in the Natural Deduction proof system for FOL, $\mathcal{N}_{\mathsf{FOL}}$.

Submit your solution by uploading it as PDF in ILIAS.

Exercise 1: Models for Quantified Formulas 2 Points

Consider the vocabulary $\mathcal{V} = (\{x, y, z\}, \{c\}, \{f\}, \{p\})$ and the following formula.

$$\varphi : \ \forall x.\exists y.p(f(c, y), x)$$

(a) Consider the model $\mathcal{M} = (\mathcal{D}, \mathcal{I})$, where $\mathcal{D} = \mathbb{Z}$ and $\mathcal{I}$ maps c to the integer number 2, f to the multiplication function ("$\cdot$"), and p to the equality relation ("$=$"). Let ρ be the variable assignment that is defined as $\rho(x) = 42$, $\rho(y) = 17$ and $\rho(z) = 23$. What is $[\![\varphi]\!]_{\mathcal{M},\rho}$? Justify your answer.

(b) Find two models $\mathcal{M}_i = (\mathcal{D}_i, \mathcal{I}_i)$ and two variable assignments ρ_i such that $[\![\varphi]\!]_{\mathcal{M}_i,\rho_i}$ is different from $[\![\varphi]\!]_{\mathcal{M},\rho}$ for $i \in \{1, 2\}$. The interpretation domain $\mathcal{D}_1$ should also be $\mathbb{Z}$, the interpretation domain $\mathcal{D}_2$ should be some different set.

Exercise 2: FOL Satisfiability 2 Points

For each of the following formulas φ_i give a model $\mathcal{M}_i$ with $[\![\varphi_i]\!]_{\mathcal{M}_i,\rho} = \textbf{true}$. You do not have to give a variable assignment ρ because for each of the formulas the evaluation to a truth value is independent of the variable assignment.

(a) $\varphi_1 : \mathit{equals}(\mathit{add}(2, 2), 5)$

(b) $\varphi_2 : \forall x.\ p(x, x)$

(c) $\varphi_3 : \exists y.\ \forall x.\ p(x, y)$

(d) $\varphi_4 : \forall x.\ (p(x, f(x)) \land \neg p(f(x), x))$

Exercise 3: Free Variables and Substitutions 2 Points

Consider the formulas φ_i and the substitutions σ_i for $i \in \{1, 2\}$ displayed below.

(i) Compute the set of free variables $\text{freevars}(\varphi_i)$ for $i \in \{1, 2\}$.

(ii) Compute the result $\varphi_i\sigma_i$ for $i \in \{1, 2\}$.

$\varphi_1 : \forall x'.\ (p(x, x') \land \exists x.\ p(x', x))$ and $\sigma_1 : \{x \mapsto a\}$
$\varphi_2 : \forall y.\ p(x, y, z)$ and $\sigma_2 : \{x \mapsto f(z), z \mapsto f(y)\}$

Exercise 4: Natural Deduction Proofs for First-Order Logic 4 Points

Use the Natural Deduction proof system for First-Order Logic $\mathcal{N}_{\mathsf{FOL}}$ to show that the following implications hold.

- $\{(\forall x.\ p(x)) \vee (\forall x.\ q(x))\} \models \forall x.\ p(x) \vee q(x)$
- $\{\exists x.\ \forall y.\ p(x, y)\} \models \forall y.\ \exists x.\ p(x, y)$

Exercise 5: Proof rules of $\mathcal{N}_{\mathsf{FOL}}$ 2 Points

In the slides, you will find the rules for the Natural Deduction proof system for First-Order Logic $\mathcal{N}_{\mathsf{FOL}}$. The proof system $\mathcal{N}_{\mathsf{FOL}}$ extends the proof system $\mathcal{N}_{\mathsf{PL}}$ by four rules.

Show that the side conditions of the rules $(I_\forall)$ and $(E_\exists)$ are necessary for the correctness of the rules. That is, for each of the rules, ignore the side condition and give an example such that the implication above the horizontal line holds, but the implication below the horizontal line does not hold.