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Hand in until 16:00 on July 14, 2021
Discussion: July 14, 2021

# Tutorial for Program Verification
## Exercise Sheet 22

In this exercise sheet we introduce well-founded relations in order to prepare Wednesday's lecture on termination.

Submit your solution by uploading it as PDF in ILIAS.

**Exercise 1: Well-Founded Relations** 3 Points

For the purpose of termination analysis, we will need the notion of a *well-founded* relation. In this exercise, we will introduce the definition and apply it to a few relations.

In the chapter on termination analysis we will work with *infinite sequences*. Analogously to a sequence of length $n$, which can be seen as a map whose domain is $\{0, \dots n-1\}$, an infinite sequence can be seen as a map whose domain are all natural numbers.

**Definition (Well-Founded Relation)** Let $X$ be a set. We call a binary relation $R \subseteq X \times X$ *well-founded* if there is no infinite sequence $x_1, x_2, \dots$ such that $(x_i, x_{i+1}) \in R$ for all $i \in \mathbb{N}$.

For each of the following relations, state if it is well-founded or not. If it is not, give an infinite sequence as a counterexample.

(a) $R_a = \{\, (x, x') \in \{\mathbf{true}, \mathbf{false}\}^2 \mid x = x' \,\}$

(b) $R_b = \{\, (x, x') \in \mathbb{N}^2 \mid x > x' \,\}$

(c) $R_c = \{\, (x, x') \in \mathbb{Z}^2 \mid x > x' \,\}$

(d) $R_d = \{\, (x, x') \in \mathbb{Q}^2 \mid x \geq 0 \text{ and } x' \geq 0 \text{ and } x > x' \,\}$

(e) $R_e = \{\, \big((x, y), (x', y')\big) \in (\mathbb{N}^2)^2 \mid x > x' \text{ or } (x = x' \text{ and } y > y') \,\}$

(f) $R_f = \{\, \big((x, y), (x', y')\big) \in (\mathbb{Z}^2)^2 \mid x' = 3 \cdot x \text{ and } y' = 2 \cdot y \text{ and } x \geq 2 \text{ and } y \geq 2\}$