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Hand in until 10:00 on July 19, 2021
Discussion: July 19, 2021

Tutorial for Program Verification
Exercise Sheet 23

Exercise 1: Termination 2 Points

In the lecture, we discussed four different properties of programs. One property was *termination* the other properties where related to termination. We provide formal definitions here. In each case, we consider a program P with a CFG $(Loc, \Delta, \ell_{\mathsf{init}}, \ell_{\mathsf{ex}})$.

(a) We say that P *can reach the exit location* if there exists a finite execution, such that the first configuration (ℓ, s) is initial, and the last configuration is (ℓ_{ex}, s') for some state s'.

(b) We say that P *can stop* if there exists a reachable configuration (ℓ, s) such that there exists no configuration (ℓ', s') and statement st with $(\ell, st, \ell') \in \Delta$ and $(s, s') \in [\![st]\!]$.

(c) We say that P *always reaches the exit location* if there exist no infinite executions, and all finite executions end in a configuration (ℓ', s') where we either have a successor (i.e., there exists a configuration (ℓ'', s'') and statement st with $(\ell', st, \ell'') \in \Delta$ and $(s', s'') \in [\![st]\!]$) or we have that ℓ' is ℓ_{ex}.

(d) We say that P *always stops* (resp. P *terminates* if there exist no infinite executions.

In this exercise, you should give programs that differentiate between these definitions. In particular, for each of the following pairs, give a program such that one definition holds but the other does not. Explain which of the definitions holds and why.

(a) *P can reach the exit location* **vs.** *P can stop*

(b) *P can stop* **vs.** *P always stops*

Exercise 2: Ranking Functions 5 Points

For each of the following programs, state whether it (always) terminates or not. If it terminates, give a ranking function for each loop in the program. If it may not terminate, give an infinite execution of the program.

```
while (x > 0) {
  while (y > 0) {
    y := y-1;

  }
  x := x-1;
  havoc y;

}
```

Listing 1: Program P_1

```
while (x > 0) {
  if (y > 0) {
    y := y-1;

  } else {
    x := x-1;
    havoc y;
  }
}
```

Listing 2: Program P_2

```
while (x > 0) {
  if (y > 0) {
    y := y-1;
    havoc x;
  } else {
    x := x-1;
    havoc y;
  }
}
```

Listing 3: Program P_3

Hint: For simple loops is often convenient to use a function whose range is $\mathbb{N}$ and the strictly greater than relation $>$ on natural numbers. For more complex loops, this is sometimes not sufficient but we can use instead a function $f : S_{V,\mu} \to \mathbb{N}_1 \times \ldots \times \mathbb{N}_n$ whose range are n-tuples of natural numbers and the *lexicographic order* $>_{\mathsf{lex}}$ that we define as follows.

> $(m_1, \ldots, m_n) >_{\mathsf{lex}} (m'_1, \ldots, m'_n)$ iff there exists $i \in \{1, \ldots n\}$ such that $m_i > m'_i$ and for all $k \in \{1, \ldots i-1\}$ the equality $m_k = m'_k$ holds

If a function with that signature together with the order $>_{\mathsf{lex}}$ is a ranking function, it is often called a *lexicographic ranking function*.

Exercise 3: Synthesis of Ranking Functions 2+2 Points

In this exercise we want to synthesize a ranking function for the program below and we want to use Farkas' Lemma in order to simplify the constraints for the SMT solver.

```
while (7*x+5*y>=4 && 7*x+3*y<=-12) {
  x := 2*x - y;
  y := 3*y + 2;
}
```

(a) Construct a transition formula for the following statement.

`assume(7*x+5*y>=4 && 7*x+3*y<=-12); x:=2*x-y; y:=3*y+2;`

(b) Write down the constraints (formula universally quantified over `x,y,x'` and `y'`) that state that the loop has a ranking function f of the form $f(x, y) = \alpha \cdot x + \beta \cdot y + \gamma$. The symbols α, β and γ are the coefficients of the ranking function for which we want to find a solution.

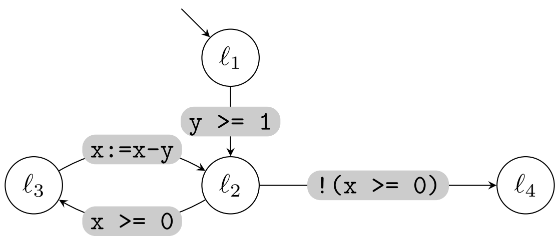
(c) Use Farkas' Lemma to transform the constraints into logically equivalent, existentially quantified, constraints that use only linear arithmetic (e.g., no multiplication of variables). This is a bonus exercise and you may submit you solution as an SMT script.

(d) Find a solution for α, β and γ. This is also a bonus exercise.

Exercise 4: Ranking Function 2 Points

Let us consider the program whose code and control-flow graph are given below.

```
assume(y >= 1);
while (x >= 0) {
  x := x - y;
}
```



The program is terminating, however there is no ranking function for the while loop. To prove termination, we introduced the following definitions in the lecture:

Definition (Loop Entry) Given a while loop `while(expr){st}` and a control-flow graph $G = (Loc, \Delta, \ell_{\mathsf{init}}, \ell_{\mathsf{ex}})$ for this while loop, we call ℓ_{init} the *entry location* of the while loop.

Definition (Ranking Function) Given a program $P = (V, \mu, st)$, a Floyd-Hoare annotation β for P, a while loop `while(expr){st}` whose loop entry is the location ℓ, and a set W together with a well-founded relation $R \subseteq W \times W$, we call a function $f : S_{V,\mu} \to W$ a *ranking function* for `while(expr){st}` and β if for each pairs of states where $s \in \{\beta(\ell)\}$ and $(s, s') \in [\![\texttt{assume expr; st}]\!]$, the relation $(f(s), f(s')) \in R$ holds.

Give a Floyd-Hoare annotation β and a function f such that f is a ranking function for β and the while loop.