Hoare Calculus

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- given precondition ϕ, θ is *inductive invariant* if:

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Annotated Programs

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- command

$$\begin{aligned} C \ ::= \ & \mathbf{skip} \mid x := e \mid C_1; C_2 \mid \mathbf{if}\ b\ \mathbf{then}\ C_1\ \mathbf{else}\ C_2 \mid \\ & \mathbf{while}\ b\ \mathbf{do}\ \{\theta\}\ C \end{aligned}$$

Example: Factorial function

- $$\begin{array}{l} \{n \geq 0\} \\ f := 1; \\ i := 1; \\ \textbf{while } i \leq n \textbf{ do } \{f = fact(i-1) \wedge i \leq n+1\} \ \{ \\ \quad f := f \times i \\ \quad i := i + 1 \\ \} \\ \{f = fact(n)\} \end{array}$$

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  not used in commands C
- interpretation of function symbol *fact* in logical model for integers (bounded or unbounded)
- axioms added in set of assertions Γ

$$
\begin{array}{l}
\mathit{fact}(0) = 1 \\
\forall n.\; n > 0 \rightarrow \mathit{fact}(n) = n \times \mathit{fact}(n-1)
\end{array}
$$

Loop Unfolding

- equivalence (proved using the semantics of programs)

$$\textbf{while } b \textbf{ do } C_0 \;\equiv\; \textbf{if } b \textbf{ then } \{C_0 \,;\, \textbf{while } b \textbf{ do } C_0\} \textbf{ else skip}$$

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- assignment

$$\frac{}{\{\psi[e/x]\}\ x := e\ \{\psi\}}$$

System $\mathcal{H}$ (2)

- sequential command $C \equiv C_1 \; ; \; C_2$

$$\frac{\{\phi\}\ C_1\ \{\phi'\} \qquad \{\phi'\}\ C\ \{\psi\}}{\{\phi\}\ C\ \{\psi\}}$$

System $\mathcal{H}$ (2)

- sequential command $C \equiv C_1 \, ; \, C_2$

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- conditional command $C \equiv \textbf{if } b \textbf{ then } C_1 \textbf{ else } C_2$

$$\frac{\{\phi \wedge b\}\ C_1\ \{\psi\} \qquad \{\phi \wedge \neg b\}\ C\ \{\psi\}}{\{\phi\}\ C\ \{\psi\}}$$

System $\mathcal{H}$ (3)

- while command $C \equiv$ **while** b **do** $\{\theta\}$ C

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- strengthen precondition, weaken postcondition

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- Hoare triple derivable in all logicals models in which implications in side condition are valid

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- inverse does not hold in general
- derivability depends on annotation with loop invariants,
  validity does not

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- more complicated inference rule for 'instantiating a Hoare triple' with auxiliary variables
- in practice, we will need adaptation only for *procedure contracts*
  which we will introduce later