Logics, Categories, and Colimits for Artificial Intelligence

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Exercise Sheet 9

Due: January 9, 2009

Exercise 9.1 (Specification morphisms)

Let SP_1, SP_2 be two specifications. Show that for any signature morphism $\sigma : \mathrm{Sig}(\mathrm{SP}_1) \to \mathrm{Sig}(\mathrm{SP}_2)$, the following are equivalent:

(a) $\sigma : \mathrm{SP}_1 \to \mathrm{SP}_2$ is a specification morphism
(b) $\mathrm{Mod}(\mathrm{SP}_2 \text{ hide } \sigma) \subseteq \mathrm{Mod}(\mathrm{SP}_1)$
(c) $\mathrm{Mod}(\mathrm{SP}_2) \subseteq \mathrm{Mod}(\mathrm{SP}_1 \text{ with } \sigma)$

Exercise 9.2 (Models of specifications)

Show that the following statements are not equivalent. Provide counterexamples for both implications.

(a) $\mathrm{Mod}(\mathrm{SP}_1) \subseteq \mathrm{Mod}(\mathrm{SP}_2 \text{ hide } \sigma)$
(b) $\mathrm{Mod}(\mathrm{SP}_1 \text{ with } \sigma) \subseteq \mathrm{Mod}(\mathrm{SP}_2)$

Exercise 9.3 (Algebraic laws for specifications)

Check which of the following algebraic laws hold:

(a) $\mathrm{SP} \text{ and } \mathrm{SP} \equiv \mathrm{SP}$
(b) $\mathrm{SP}_1 \text{ and } \mathrm{SP}_2 \equiv \mathrm{SP}_2 \text{ and } \mathrm{SP}_1$
(c) $(\mathrm{SP} \text{ with } \sigma_1) \text{ with } \sigma_2 \equiv \mathrm{SP} \text{ with } \sigma_2 \circ \sigma_1$
(d) $(\mathrm{SP}_1 \text{ and } \mathrm{SP}_2) \text{ with } \sigma \equiv (\mathrm{SP}_1 \text{ with } \sigma) \text{ and } (\mathrm{SP}_2 \text{ with } \sigma)$
(e) $(\mathrm{SP} \text{ hide } \sigma_2) \text{ hide } \sigma_1 \equiv \mathrm{SP} \text{ hide } \sigma_2 \circ \sigma_1$
(f) $(\mathrm{SP}_1 \text{ and } \mathrm{SP}_2) \text{ hide } \sigma \equiv (\mathrm{SP}_1 \text{ hide } \sigma) \text{ and } (\mathrm{SP}_2 \text{ hide } \sigma)$
(g) $(\mathrm{SP} \text{ with } \sigma) \text{ hide } \sigma \equiv \mathrm{SP}$
(h) $(\mathrm{SP} \text{ hide } \sigma) \text{ with } \sigma \equiv \mathrm{SP}$

Exercise 9.4 (Christmas bonus problem: existence of Santa Clause I)

Explain what is wrong with the following proof of the existence of Santa Clause.

Recall the ∃-introduction rule:

$$\frac{\varphi(t)}{\exists x : s.\varphi(x)}$$

Theorem. Santa Clause exists.
Proof. Assume to the contrary that Santa Clause does not exist. By ∃-introduction, there exists something that does not exist. This is a contradiction. Hence, the assumption that Santa Clause does not exist must be wrong. Thus, Santa Clause exists. □

Exercise 9.5 (Christmas bonus problem: existence of Santa Clause II)

So, the existence proof in the last exercise was flawed. But I have another proof. What about this one?

Let c be the set $\{x \mid \text{if } x \in x, \text{ then Santa Clause exists}\}$.
Now, assume that $c \in c$. We know that $c \in c$ iff "if $c \in c$, then Santa Clause exists". Therefore, we may assert that "if $c \in c$, then Santa Clause exists". Thus by modus ponens, Santa Clause exists.
However, I have just proven that "if $c \in c$, then Santa Clause exists". Thus, $c \in c$, and therefore Santa Clause exists. □

Exercise 9.6 (Christmas bonus problem: all reindeers have the same color)

So I could not convince you that Santa Clause exists. Can I convince you that all reindeers have the same color? The following proof should assure you of that claim, shouldn't it?

Theorem. Any number of reindeers have the same color.
Proof. By induction.

Basis: one reindeer has the same color (obviously!).

Inductive step: suppose that any collection of n reindeers has the same color. We need to show that $n + 1$ reindeers have the same color, too. By induction hypothesis, the first n reindeers have the same color. Take out the last reindeer of these and replace it with the $n+1$st. Again by induction hypothesis, these have the same color. Hence, all $n + 1$ reindeers have the same color. □

The exercise sheets may and should be worked on in groups of two (2) students. Please write both names on your solution.